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Estimation of parameters of boundary value problems for linear ordinary differential equations with uncertain data

Contents

Abstract

In this paper we construct optimal, in certain sense, estimates of values of linear functionals on solutions to two-point boundary value problems (BVPs) for systems of linear first-order ordinary differential equations from observations which are linear transformations of the same solutions perturbed by additive random noises. It is assumed here that right-hand sides of equations and boundary data as well as statistical characteristics of random noises in observations are not known and belong to certain given sets in corresponding functional spaces. This leads to the necessity of introducing minimax statement of an estimation problem when optimal estimates are defined as linear, with respect to observations, estimates for which the maximum of mean square error of estimation taken over the above-mentioned sets attains minimal value. Such estimates are called minimax estimates.

We establish that the minimax estimates are expressed via solutions of some systems of differential equations of special type.

Similar estimation problems for solutions of BVPs for linear differential equations of order n with general boundary conditions are considered.

We also elaborate minimax estimation methods under incomplete data of unknown right-hand sides of equations and boundary data and obtain representations for the corresponding minimax estimates.

In all the cases estimation errors are determined.

Introduction

Minimax estimation is studied in a big number of works; one may refer e.g. to [9]–[12] and the bibliography therein.

Let us formulate a general approach to the problem. If a state of a system is described by a linear ordinary differential equation

$$\frac{dx(t)}{dt} = Ax(t) + Bv_1(t), \quad x(t_0) = x_0,$$

and a function $y(t)$ is observed in a time interval $[t_0, T]$, where $y(t) = Hx(t) + v_2(t)$, $x(t) \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and A, B, H are known matrices, the minimax estimation problem consists in the most accurate determination of a function $x(t)$ at the "worst" realization of unknown quantities $(x_0, v_1(\cdot), v_2(\cdot))$ taken from a certain set. N.N. Krasovskii was the first who stated this problem in [10]. Under different constraints imposed on function $v_2(t)$ and for known function $v_1(t)$ he proposed various methods of estimating inner products $(a, x(T))$ in the class of operations linear with respect to observations that minimize the maximal error. Later these estimates were called minimax a priori estimates (see [10], [12]).

Fundamental results concerning estimation under uncertainties were obtained by A. B. Kurzhanskii (see [12], [13]).

The duality principle elaborated in [10], [12], and [9] proved its efficiency for the determination of minimax estimates [9]. According to this principle, finding minimax a priori estimates can be reduced to a certain problem of optimal control of a system; this approach enabled one to obtain, under certain restrictions, recurrent equations, namely, the minimax Kalman–Bucy filter (see [9]).

In this work we consider the problems of minimax estimation of solutions to two-point boundary value problems (BVPs) for systems of linear first-order ordinary differential equations. We find general form of the minimax estimates of solutions from observations on an interval and determine estimation errors.

In the second part of the work (section 6 and 7) we formulate and solve the problems of estimation under incomplete data of the values of linear functionals from solutions and right-hand sides of linear

differential equations of order n with general boundary conditions. Additional difficulties that arise in the course of the analysis of these estimation problems are connected with (i) the necessity of imposing certain solvability conditions on the data (right-hand sides of the equations and boundary conditions) and (ii) that their solutions are determined up to solutions of the corresponding homogeneous problems.

1 Preliminaries and auxiliary results

Assume that it is given a vector-function $f(t) = (f_1(t), f_2(t) \dots f_n(t))^T$ with the components belonging to space $L^2(0,T)$ and vectors $f_0 = (f_1^{(0)}, f_2^{(0)}, \dots, f_m^{(0)})^T \in \mathbb{R}^m$ and $f_1 = (f_1^{(1)}, f_2^{(1)}, \dots, f_{n-m}^{(1)})^T \in \mathbb{R}^{n-m}$. Consider the following BVP: find a vector-function $\varphi(t) = (\varphi_1(t), \varphi_2(t) \dots, \varphi_n(t))^T \in H^1(0,T)^n$ that satisfies a system of linear first-order ordinary differential equations

$$\frac{d\varphi(t)}{dt} + A\varphi(t) = f(t), \quad t \in (0,T), \tag{1.1}$$

almost everywhere on an interval $(0,T)$ and the boundary conditions

$$B_0\varphi(0) = f_0, \quad B_1\varphi(T) = f_1 \tag{1.2}$$

at the points 0 and T. Here $A = A(t)$ is an $n \times n$ matrix with the entries $a_{ij} = a_{ij}(t)$ continuous on $[0,T]$, $\frac{d\varphi(t)}{dt} = (\frac{d\varphi_1(t)}{dt}, \frac{d\varphi_2(t)}{dt} \dots, \frac{d\varphi_n(t)}{dt})^T$, $B_0 = \{b_{rs}^{(0)}\}$, $r = \overline{1,m}$, $s = \overline{1,n}$, and $B_1 = \{b_{rs}^{(1)}\}$, $r = \overline{1,n-m}$, $s = \overline{1,n}$, are $m \times n$ and $(n-m) \times n$ matrices of rank m and $n-m$, respectively, T denotes transposition, and $H^1(a,b)$ is a space of functions absolutely continuous on $[a,b]$ for which the derivative that exists almost everywhere on (a,b) belongs to space $L^2(a,b)$.

The problem of finding a function $\varphi(t)$ that satisfies on $(0,T)$ the equation

$$\frac{d\varphi(t)}{dt} + A\varphi(t) = 0, \tag{1.3}$$

and the boundary conditions

$$B_0\varphi(0) = 0, \quad B_1\varphi(T) = 0 \tag{1.4}$$

will be called the homogeneous BVP corresponding to BVP (1.1), (1.2).

The solution $\varphi(t) \equiv 0$ to homogeneous BVP (1.3), (1.4) is called the trivial solution.

BVP (1.1), (1.2) can be written in a scalar form:

$$\begin{array}{lcl} \varphi_1'(t) + a_{11}\varphi_1(t) + a_{12}\varphi_2(t) + \dots + a_{1n}\varphi_n(t) & = & f_1(t), \\ \varphi_2'(t) + a_{21}\varphi_1(t) + a_{22}\varphi_2(t) + \dots + a_{2n}\varphi_n(t) & = & f_1(t), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \dots & \dots, \\ \varphi_n'(t) + a_{n1}\varphi_1(t) + a_{n2}\varphi_2(t) + \dots + a_{nn}\varphi_n(t) & = & f_n(t), \end{array} \tag{1.5}$$

$$\begin{aligned} U_i(\varphi) &:= \sum_{q=1}^n b_{iq}^{(0)} \varphi_q(0) = f_i^{(0)}, \quad i = \overline{1,m}, \\ U_{m+i}(\varphi) &:= \sum_{q=1}^n b_{iq}^{(1)} \varphi_q(T) = f_i^{(1)}, \quad i = \overline{1,n-m}. \end{aligned} \tag{1.6}$$

Let

$$\varphi^{(i)}(t) = (\varphi_1^{(i)}(t), \varphi_2^{(i)}(t) \dots, \varphi_n^{(i)}(t))^T, \quad i = \overline{1,n}, \tag{1.7}$$

be a fundamental system of solutions to (1.3) (for the definition, see e.g. [8] p. 179). Then the solutions to (1.3), (1.4) have the form

$$\varphi(t) = c_1\varphi^{(1)}(t) + c_2\varphi^{(2)}(t) + \dots + c_n\varphi^{(n)}(t),$$

where, by virtue of (1.4), constants $c_1, c_2, \dots, c_n$ must be such that

$$\begin{array}{rcl} c_1U_1(\varphi^{(1)}) + c_2U_1(\varphi^{(2)}) + \dots + c_nU_1(\varphi^{(n)}) & = & 0, \\ c_1U_2(\varphi^{(1)}) + c_2U_2(\varphi^{(2)}) + \dots + c_nU_2(\varphi^{(n)}) & = & 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \dots & .., \\ c_1U_n(\varphi^{(1)}) + c_2U_n(\varphi^{(2)}) + \dots + c_nU_n(\varphi^{(n)}) & = & 0. \end{array} \tag{1.8}$$

Thus, if the matrix

$$\begin{pmatrix} U_1(\varphi^{(1)}) & U_1(\varphi^{(2)}) & \cdots & U_1(\varphi^{(n)}) \\ U_2(\varphi^{(1)}) & U_2(\varphi^{(2)}) & \cdots & U_2(\varphi^{(n)}) \\ \cdots & \cdots & \cdots & \cdots \\ U_n(\varphi^{(1)}) & U_n(\varphi^{(2)}) & \cdots & U_n(\varphi^{(n)}) \end{pmatrix} \tag{1.9}$$

has rank n, the homogeneous BVP has only the trivial solution. The inverse statement is also valid: if the homogeneous BVP has only the trivial solution then the rank of matrix (1.9) equals n. Indeed, following the reasoning that can be found e.g. in [4], assume that the rank of this matrix is $r < n$; then system (1.8) would have $n - r$ linearly independent solutions $c^{(i)} = (c_1^{(i)}, \dots, c_n^{(i)})^T$, $i = \overline{1, n-r}$ (see e.g. [7], p. 85). Let us show that if this assumption holds then the functions

$$\tilde{\varphi}^{(i)}(x) = c_1^{(i)}\varphi^{(1)}(x) + \dots + c_n^{(i)}\varphi^{(n)}(x) \quad i = \overline{1, n-r}, \tag{1.10}$$

satisfying conditions (1.3), (1.4) will be linearly independent; that is, the equality

$$\sum_{i=1}^{n-r} \alpha_i \tilde{\varphi}^{(i)}(x) = 0 \tag{1.11}$$

is fulfilled only at $\alpha_i = 0$, $i = \overline{1, n-r}$. Substituting (1.10) into (1.11), we have

$$\sum_{i=1}^{n-r} \alpha_i \sum_{k=1}^{n} c_k^{(i)} \varphi^{(k)}(x) = \sum_{k=1}^{n} \varphi^{(k)}(x) \sum_{i=1}^{n-r} \alpha_i c_k^{(i)} = \sum_{k=1}^{n} \beta_k \varphi^{(k)}(x) = 0,$$

where $\beta_k = \sum_{i=1}^{n-r} \alpha_i c_k^{(i)}$. However, vector-functions $\varphi^{(k)}(x)$, $k = \overline{1, n}$, are linearly independent; therefore, $\beta_k = 0$, $k = \overline{1, n}$, or $\sum_{i=1}^{n-r} \alpha_i c_k^{(i)} = 0$, $k = \overline{1, n}$. Then all $\alpha_i = 0$, $i = \overline{1, n-r}$, because vectors $c^{(i)} = (c_1^{(i)}, \dots, c_n^{(i)})^T$, $i = \overline{1, n-r}$, are linearly independent. Next, linear independence of functions (1.10) satisfying (1.3), (1.4), contradicts the assumption that BVP (1.3), (1.4) has only the trivial solution which means that the rank of matrix (1.9) is n.

Assume in what follows that homogeneous BVP (1.3), (1.4) corresponding to BVP (1.1), (1.2) has only the trivial solution. Show that under this assumption, initial BVP (1.1), (1.2) is uniquely solvable at any right-hand sides $f(t) = (f_1(t), f_2(t) \dots f_n(t))^T$, $f_0 = (f_1^{(0)}, f_2^{(0)}, \dots, f_m^{(0)})^T \in \mathbb{R}^m$, and $f_1 = (f_1^{(1)}, f_2^{(1)}, \dots, f_{n-m}^{(1)})^T \in \mathbb{R}^{n-m}$.

Indeed, let (1.7) be a fundamental system of solutions to homogeneous system (1.3) and $\varphi^{(0)}(t) = (\varphi_1^{(0)}(t), \varphi_2^{(0)}(t) \dots, \varphi_n^{(0)}(t))^T$ a particular solution to (1.1). Then the general solution to system (1.1) or to equivalent system (1.5) has the form

$$\varphi(t) = c_1\varphi^{(1)}(t) + c_2\varphi^{(2)}(t) + \dots + c_n\varphi^{(n)}(t) + \varphi^{(0)}(t),$$

where $c_i = \text{const}$. This solution satisfies conditions (1.2) or equivalent conditions (1.6) if coefficients

$c_i,\ i=\overline{1,n}$, satisfy the system of linear algebraic equations

$$\begin{array}{rcl} c_1U_1(\varphi^{(1)})+c_2U_1(\varphi^{(2)})+\cdots+c_nU_1(\varphi^{(n)}) & = & f_1^{(0)}-U_1(\varphi^{(0)}),\\ c_1U_2(\varphi^{(1)})+c_2U_2(\varphi^{(2)})+\cdots+c_nU_2(\varphi^{(n)}) & = & f_2^{(0)}-U_2(\varphi^{(0)}),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots,\\ c_1U_m(\varphi^{(1)})+c_2U_m(\varphi^{(2)})+\cdots+c_nU_m(\varphi^{(n)}) & = & f_m^{(0)}-U_m(\varphi^{(0)}),\\ c_1U_{m+1}(\varphi^{(1)})+c_2U_{m+1}(\varphi^{(2)})+\cdots+c_nU_{m+1}(\varphi^{(n)}) & = & f_1^{(1)}-U_{m+1}(\varphi^{(0)}),\\ c_1U_{m+2}(\varphi^{(1)})+c_2U_{m+2}(\varphi^{(2)})+\cdots+c_nU_{m+2}(\varphi^{(n)}) & = & f_2^{(1)}-U_{m+2}(\varphi^{(0)}),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots,\\ c_1U_n(\varphi^{(1)})+c_2U_n(\varphi^{(2)})+\cdots+c_nU_n(\varphi^{(n)}) & = & f_{n-m}^{(1)}-U_n(\varphi^{(0)}). \end{array} \tag{1.12}$$

The rank of matrix (1.9) of this system is n because homogeneous BVP (1.3), (1.4) has only the trivial solution. Therefore, system (1.12) and, consequently, BVP (1.1), (1.2), have the unique solution. We have proved the following

Theorem 1.1. *Inhomogeneous BVP (1.1), (1.2) is uniquely solvable if and only if the corresponding homogeneous BVP (1.3), (1.4) has only the trivial solution.*

Formulate the notion of a BVP conjugate to (1.1), (1.2). To this end, introduce the following designations: E_k is the $k\times k$ unit matrix; $O_{k,r}$ is the $k\times r$ null matrix; $B_{01}=\{b^{(0)}_{ri_k}\}$, $r=\overline{1,m}$, $k=\overline{1,m}$, is a square nondegenerate $m\times m$ submatrix of the matrix $B_0=\{b^{(0)}_{rs}\}$, $r=\overline{1,m}$, $s=\overline{1,n}$; $B_{02}=\{b^{(0)}_{rj_l}\}$, $r=\overline{1,m}$, $l=\overline{1,n-m}$, is an $m\times(n-m)$ submatrix of B_0 obtained as a result of deleting in B_0 all columns of matrix B_{01} (so that $\{j_1,\dots,j_{n-m}\}=\{1,\dots,n\}\setminus\{i_1,\dots,i_m\}$); $\hat{B}_0=(-B_{02}^T(B_{01}^T)^{-1},E_{n-m})$ is an $(n-m)\times n$ matrix such that its i_kth column equals kth column of matrix $-B_{02}^T(B_{01}^T)^{-1}$ (its size is $(n-m)\times m$), $k=\overline{1,m}$, and j_lth column equals lth column of matrix E_{n-m}, $l=\overline{1,n-m}$; $\bar{B}_0=((B_{01}^T)^{-1},O_{m,n-m})$ is an $m\times n$ matrix such that its i_kth column equals $k-$th column of matrix $(B_{01}^T)^{-1}$, $k=\overline{1,m}$, and j_lth column equals lth column of matrix $O_{m,n-m}$, $l=\overline{1,n-m}$; $\tilde{B}_0=(O_{n-m,m},E_{n-m})$ is an $(n-m)\times n$ matrix such that its i_kth column equals kth column of matrix $O_{n-m,m}$, $k=\overline{1,m}$, and j_lth column equals lth column of matrix E_{n-m}, $l=\overline{1,n-m}$.

Introduce more similar notations: $B_{11}=\{b^{(1)}_{ri'_k}\}$, $r=\overline{1,n-m}$, $k=\overline{1,n-m}$, is a square nondegenerate $(n-m)\times(n-m)$ submatrix of the matrix $B_1=\{b^{(1)}_{rs}\}$, $r=\overline{1,n-m}$, $s=\overline{1,n}$; $B_{12}=\{b^{(1)}_{rj'_l}\}$, $r=\overline{1,n-m}$, $l=\overline{1,m}$, is a $(n-m)\times m$ submatrix of the matrix B_1 obtained as a result of deleting in B_1 all columns of matrix B_{11} (so that $\{j'_1,\dots,j'_m\}=\{1,\dots,n\}\setminus\{i'_1,\dots,i'_{n-m}\}$); $\hat{B}_1=(-B_{12}^T(B_{11}^T)^{-1},E_m)$ is an $m\times n$ matrix such that its i'_kth column equals kth column of matrix $-B_{12}^T(B_{11}^T)^{-1}$ (the size of the latter is $m\times(n-m)$), $k=\overline{1,n-m}$, and j'_lth column equals lth column of matrix E_m, $l=\overline{1,m}$; $\bar{B}_1=((B_{11}^T)^{-1},O_{n-m,m})$ is an $(n-m)\times n$ matrix such that its i'_kth column equals kth column of matrix $(B_{11}^T)^{-1}$, $k=\overline{1,n-m}$, and j'_lth column equals lth column of matrix $O_{n-m,m}$, $l=\overline{1,m}$; $\tilde{B}_1=(O_{m,n-m},E_m)$ is an $m\times n$ matrix such that its i'_kth column equals kth column of matrix $O_{m,n-m}$, $k=\overline{1,n-m}$, and j'_lth column equals lth column of matrix E_m, $l=\overline{1,m}$.

By

$$(u,v)_N=\sum_{i=1}^{N}u_iv_i$$

we will denote the inner product of vectors $u=(u_1,\dots,u_N)^T$ and $v=(v_1,\dots,v_N)^T$ in the Euclidean space $\mathbb{R}^N$. Set

$$L=\frac{d}{dt}+A,$$

then

$$L\varphi(t)=\frac{d\varphi(t)}{dt}+A\varphi(t).$$

Calculate the inner product of both sides of the latter equality and the vector-function $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))^T$ and integrate the result from 0 to T to obtain

$$\int_0^T (L\varphi(t), \psi(t))_n dt = \int_0^T \left(\frac{d\varphi(t)}{dt} + A\varphi(t), \psi(t)\right)_n dt$$

$$= \int_0^T \left(\frac{d\varphi(t)}{dt}, \psi(t)\right)_n dt + \int_0^T (A\varphi(t), \psi(t))_n \, dt$$

$$= \int_0^T \sum_{i=1}^n \frac{d\varphi_i(t)}{dt}\psi_i(t)dt + \int_0^T \sum_{i=1}^n \sum_{j=1}^n a_{ij}\varphi_j(t)\psi_i(t)dt$$

$$= \sum_{i=1}^n \varphi_i(t)\psi_i(t)\,|_0^T - \int_0^T \sum_{i=1}^n \frac{d\psi_i(t)}{dt}\varphi_i(t)dt + \int_0^T \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\psi_i(t)\varphi_j(t)\right) dt$$

$$= (\varphi(T), \psi(T))_n - (\varphi(0), \psi(0))_n - \int_0^T \left(\frac{d\psi(t)}{dt}, \varphi(t)\right)_n dt + \int_0^T \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}\psi_j(t)\varphi_i(t)\right) dt$$

$$= (\varphi(T), \psi(T))_n - (\varphi(0), \psi(0))_n + \int_0^T \left(-\frac{d\psi(t)}{dt}, \varphi(t)\right)_n dt + \int_0^T \left(\varphi(t), A^T\psi(t)\right)_n dt$$

$$= (\varphi(T), \psi(T))_n - (\varphi(0), \psi(0))_n + \int_0^T (\varphi(t), L^*\psi(t))_n \,, \tag{1.13}$$

where the differential operator

$$L^* = -\frac{d}{dt} + A^T$$

will be called formally conjugate to operator L.

Let us show that the integrands in (1.13) can be represented as

$$(\psi(T), \varphi(T))_n - (\psi(0), \varphi(0))_n = (\bar{B}_1\psi(T), B_1\varphi(T))_{n-m} + (\hat{B}_1\psi(T), \tilde{B}_1\varphi(T))_m$$
$$- (\bar{B}_0\psi(0), B_0\varphi(0))_m - (\hat{B}_0\psi(0), \tilde{B}_0\varphi(0))_{n-m}. \tag{1.14}$$

Note first that

$$B_0\varphi(0) = \begin{pmatrix} \sum_{q=1}^n b_{1q}^{(0)}\varphi_q(0) \\ \vdots \\ \sum_{q=1}^n b_{nq}^{(0)}\varphi_q(0) \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^n b_{1i_k}^{(0)}\varphi_{i_k}(0) + \sum_{l=1}^{n-m} b_{1j_l}^{(0)}\varphi_{j_l}(0) \\ \vdots \\ \sum_{k=1}^n b_{mi_k}^{(0)}\varphi_{i_k}(0) + \sum_{l=1}^{n-m} b_{mj_l}^{(0)}\varphi_{j_l}(0) \end{pmatrix}$$

$$= B_{01}\varphi_1^{(0)}(0) + B_{02}\varphi_2^{(0)}(0),$$

where

$$\varphi_1^{(0)}(0) := \begin{pmatrix} \varphi_{i_1}(0) \\ \vdots \\ \varphi_{i_m}(0) \end{pmatrix}, \quad \varphi_2^{(0)}(0) := \begin{pmatrix} \varphi_{j_1}(0) \\ \vdots \\ \varphi_{j_{n-m}}(0) \end{pmatrix}.$$

Then $\varphi_1^{(0)}(0) = B_{01}^{-1}B_0\varphi(0) - B_{01}^{-1}B_{02}\varphi_2^{(0)}(0)$, and

$$(\psi(0), \varphi(0))_n = (\psi_1^{(0)}(0), \varphi_1^{(0)}(0))_m + (\psi_2^{(0)}(0), \varphi_2^{(0)}(0))_{n-m}$$

$$= (\psi_1^{(0)}(0), B_{01}^{-1}B_0\varphi(0))_m - (\psi_1^{(0)}(0), B_{01}^{-1}B_{02}\varphi_2^{(0)}(0))_m + (\psi_2^{(0)}(0), \varphi_2^{(0)}(0))_{n-m}$$

$$= (\bar{B}_0\psi(0), B_0\varphi(0))_m - (B_{02}^T(B_{01}^T)^{-1}\psi_1^{(0)}(0), \varphi_2^{(0)}(0))_{n-m} + (\psi_2^{(0)}(0), \varphi_2^{(0)}(0))_{n-m}$$

$$= (\bar{B}_0\psi(0), B_0\varphi(0))_m + ((-B_{02}^T(B_{01}^T)^{-1}, 0)\psi(0), \varphi_2^{(0)}(0))_{n-m}$$
$$+\big((0, E_{n-m})\psi(0), \varphi_2^{(0)}(0)\big)_{n-m},$$

where $\psi_1^{(0)}(0)$ and $\psi_2^{(0)}(0)$ are vectors composed of components of vector $\psi(0)$ with the numbers equal to the numbers of components of vectors $\varphi_1^{(0)}(0)$ and $\varphi_2^{(0)}(0)$, respectively. Taking into account that

$$(-B_{02}^T(B_{01}^T)^{-1}, O_{n-m,n-m}) + (O_{n-m,m}, E_{n-m}) = (-B_{02}^T(B_{01}^T)^{-1}, E_{n-m}) = \hat{B}_0,$$

we have

$$(\psi(0), \varphi(0))_n = (\bar{B}_0\psi(0), B_0\varphi(0))_m + (\hat{B}_0\psi(0), \tilde{B}_0\varphi(0))_{n-m}.$$

Thus

$$(\psi(T), \varphi(T))_n = (\bar{B}_1\psi(T), B_1\varphi(T))_{n-m} + (\hat{B}_1\psi(T), \tilde{B}_1\varphi(T))_m.$$

These two equalities yield representation (1.14); using the latter and (1.13), we obtain

$$\int_0^T (L\varphi(t), \psi(t))_n dt = (\bar{B}_1\psi(T), B_1\varphi(T))_{n-m} + (\hat{B}_1\psi(T), \tilde{B}_1\varphi(T))_m$$
$$- (\bar{B}_0\psi(0), B_0\varphi(0))_m - (\hat{B}_0\psi(0), \tilde{B}_0\varphi(0))_{n-m} + \int_0^T (\varphi(t), L^*\psi(t))_n . \qquad (1.15)$$

In order to write the sum of the first four terms on the right-hand side of (1.15) in a scalar form, introduce the following notations:

$$\begin{pmatrix} U_{n+1}(\varphi) \\ \vdots \\ U_{2n-m}(\varphi) \end{pmatrix} := \tilde{B}_0\varphi(0) = \begin{pmatrix} \sum_{q=1}^n \tilde{b}_{1q}^{(0)}\varphi_q(0) \\ \vdots \\ \sum_{q=1}^n \tilde{b}_{n-m,q}^{(0)}\varphi_q(0) \end{pmatrix}, \qquad (1.16)$$

$$\begin{pmatrix} U_{2n-m+1}(\varphi) \\ \vdots \\ U_{2n}(\varphi) \end{pmatrix} := \tilde{B}_1\varphi(T) = \begin{pmatrix} \sum_{q=1}^n \tilde{b}_{1q}^{(1)}\varphi_q(T) \\ \vdots \\ \sum_{q=1}^n \tilde{b}_{m,q}^{(1)}\varphi_q(T) \end{pmatrix}, \qquad (1.17)$$

$$\begin{pmatrix} V_{2n}(\psi) \\ \vdots \\ V_{2n-m+1}(\psi) \end{pmatrix} := \bar{B}_0\psi(0) = \begin{pmatrix} \sum_{q=1}^n \bar{b}_{1q}^{(0)}\psi_q(0) \\ \vdots \\ \sum_{q=1}^n \bar{b}_{m,q}^{(0)}\psi_q(0) \end{pmatrix}, \qquad (1.18)$$

$$\begin{pmatrix} V_{2n-m}(\psi) \\ \vdots \\ V_{n+1}(\psi) \end{pmatrix} := \bar{B}_1\psi(T) = \begin{pmatrix} \sum_{q=1}^n \bar{b}_{1q}^{(1)}\psi_q(T) \\ \vdots \\ \sum_{q=1}^n \bar{b}_{n-m,q}^{(1)}\psi_q(T) \end{pmatrix}, \qquad (1.19)$$

$$\begin{pmatrix} V_{n}(\psi) \\ \vdots \\ V_{m+1}(\psi) \end{pmatrix} := \hat{B}_0\psi(0) = \begin{pmatrix} \sum_{q=1}^n \hat{b}_{1q}^{(0)}\psi_q(0) \\ \vdots \\ \sum_{q=1}^n \hat{b}_{n-m,q}^{(0)}\psi_q(0) \end{pmatrix}, \qquad (1.20)$$

$$\begin{pmatrix} V_{m}(\psi) \\ \vdots \\ V_{1}(\psi) \end{pmatrix} := \hat{B}_1\psi(T) = \begin{pmatrix} \sum_{q=1}^n \hat{b}_{1q}^{(1)}\psi_q(T) \\ \vdots \\ \sum_{q=1}^n \hat{b}_{m,q}^{(1)}\psi_q(T) \end{pmatrix}. \qquad (1.21)$$

The equality (1.15) can be written as

$$\int_0^T (L\varphi(t), \psi(t))_n dt - \int_0^T (\varphi(t), L^*\psi(t))_n$$

$$
\begin{aligned}
&= -U_1(\varphi)V_{2n}(\psi) - U_2(\varphi)V_{2n-1}(\psi) - \cdots - U_m(\varphi)V_{2n-m+1}(\psi)\\
&+U_{m+1}(\varphi)V_{2n-m}(\psi) + U_{m+2}(\varphi)V_{2n-m-1}(\psi) + \cdots + U_n(\varphi)V_{n+1}(\psi)\\
&-U_{n+1}(\varphi)V_n(\psi) - U_{n+2}(\varphi)V_{n-1}(\psi) - \cdots - U_{2n-m}(\varphi)V_{m+1}(\psi)\\
&+ U_{2n-m+1}(\varphi)V_m(\psi) + U_{2n-m+2}(\varphi)V_{m-1}(\psi) + \cdots + U_{2n}(\varphi)V_1(\psi).
\end{aligned} \tag{1.22}
$$

Now we can introduce the notion of the conjugate BVP.

Corollary 1.1. *The homogeneous BVP*

$$L^*\psi(t) = 0, \quad t \in (0, T), \tag{1.23}$$

$$\hat{B}_0\psi(0) = 0, \quad \hat{B}_1\psi(T) = 0, \tag{1.24}$$

is called conjugate to homogeneous BVP (1.3), (1.4).

Corollary 1.2. *The inhomogeneous BVP*

$$L^*\psi(t) = \tilde{f}(t), \quad t \in (0, T), \tag{1.25}$$

$$\hat{B}_0\psi(0) = \tilde{f}_0, \quad \hat{B}_1\psi(T) = \tilde{f}_1, \tag{1.26}$$

is called conjugate to inhomogeneous BVP (1.1), (1.2).

Using designations (1.6) and (1.16)–(1.21), we can write BVP (1.23), (1.24) conjugate to BVP (1.3), (1.4) in the scalar form:

$$
\begin{array}{rcl}
-\psi_1'(t) + a_{11}\psi_1(t) + a_{21}\psi_2(t) + \cdots + a_{n1}\psi_n(t) & = & 0,\\
-\psi_2'(t) + a_{12}\psi_1(t) + a_{22}\psi_2(t) + \cdots + a_{n2}\psi_n(t) & = & 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots,\\
-\psi_n'(t) + a_{1n}\psi_1(t) + a_{2n}\psi_2(t) + \cdots + a_{nn}\psi_n(t) & = & 0,
\end{array} \tag{1.27}
$$

$$V_i(\psi) := 0, \quad i = \overline{1, n}. \tag{1.28}$$

Let $z^{(1)}(t)$, $z^{(2)}(t)$, $\ldots$, $z^{(n)}(t)$ is a fundamental system of solutions to the homogeneous system $L^*\psi(t) = 0$. Show that the rank of matrix

$$
\begin{pmatrix}
V_1(z^{(1)}) & V_1(z^{(2)}) & \cdots & V_1(z^{(n)})\\
V_2(z^{(1)}) & V_2(z^{(2)}) & \cdots & V_2(z^{(n)})\\
\cdots & \cdots & \cdots & \cdots\\
V_n(z^{(1)}) & V_n(z^{(2)}) & \cdots & V_n(z^{(n)})
\end{pmatrix} \tag{1.29}
$$

equals n. Assume that it is wrong and the rank of matrix (1.29) is $r < n$. Every solution of the equation $L^*\psi(t) = 0$ and, in particular, of homogeneous BVP (1.27), (1.28) has the form

$$\psi(t) = c_1 z^{(1)}(t) + \cdots + c_n z^{(n)}(t),$$

where $c_k = \text{const}$, $k = \overline{1, n}$. Substituting the latter into (1.28), we obtain a homogeneous linear equation system

$$
\begin{array}{rcl}
V_1(\psi) = c_1V_1(z^{(1)}) + \cdots + c_nV_1(z^{(n)}) & = & 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdot,\\
V_n(\psi) = c_1V_n(z^{(1)}) + \cdots + c_nV_n(z^{(n)}) & = & 0
\end{array} \tag{1.30}
$$

with respect to constants c_k, $k = \overline{1, n}$. Since the rank of the system matrix equals r and $r < n$, system (1.30) has $n-r$ linearly independent solutions $c^{(i)} = (c_1^{(i)}, \ldots, c_n^{(i)})^T$, $i = \overline{1, n-r}$; therefore, the functions

$$\psi^{(i)}(t) = c_1^{(i)} z^{(1)}(t) + \cdots + c_n z^{(n)}(t),$$

which solve conjugate homogeneous BVP (1.27), (1.28) will be linearly independent (in line with the reasoning on p. 5).

If now $\psi(t)$ is a solution to homogeneous BVP (1.27), (1.28), then, if we set $\varphi(t) = \varphi^{(i)}(t)$, where $\varphi^{(i)}(t)$, $i = \overline{1,n}$ is the fundamental system of solutions to the homogeneous equation $L\varphi(t) = 0$, the integrals in (1.22) vanish. Also, $V_1(\psi) = V_2(\psi) = \ldots = V_n(\psi) = 0$, and therefore, (1.22) takes the form

$$\begin{array}{rcl} U_1(\varphi^{(1)})(-V_{2n}(\psi))+\cdots+U_m(\varphi^{(1)})(-V_{2n-m+1}(\psi))+U_{m+1}(\varphi^{(1)})V_{2n-m}(\psi)+\cdots+U_n(\varphi^{(1)})V_{n+1}(\psi) & = & 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdot, \\ U_1(\varphi^{(n)})(-V_{2n}(\psi))+\cdots+U_m(\varphi^{(n)})(-V_{2n-m+1}(\psi))+U_{m+1}(\varphi^{(n)})V_{2n-m}(\psi)+\cdots+U_n(\varphi^{(n)})V_{n+1}(\psi) & = & 0. \end{array} \tag{1.31}$$

This system has $n-r$ linearly independent solutions

$$-V_{2n}(\psi^{(i)}),\ \cdots,\ -V_{2n-m+1}(\psi^{(i)}),\ V_{2n-m}(\psi^{(i)}),\ \cdots,\ V_{n+1}(\psi^{(i)}) \quad (i = 1, 2, \ldots, n-r), \tag{1.32}$$

Indeed, if we assume their linear dependence, then the rank of matrix

$$\begin{pmatrix} V_{n+1}(\psi^{(1)}) & \cdots & V_{2n-m}(\psi^{(1)}) & -V_{2n-m+1}(\psi^{(1)}) & \cdots & -V_{2n}(\psi^{(1)}) \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots \\ V_{n+1}(\psi^{(n-r)}) & \cdots & V_{2n-m}(\psi^{((n-r))}) & -V_{2n-m+1}(\psi^{((n-r))}) & \cdots & -V_{2n}(\psi^{((n-r))}) \end{pmatrix} \tag{1.33}$$

will be less than $n-r$. However, since the rank of (1.33) equals the maximum number of its linearly independent rows, there exist numbers $a_1, \ldots, a_{n-r}$, such that at least one of them is nonzero and

$$a_1 V_i(\psi^{(1)}) + a_2 V_i(\psi^{(2)}) + \cdots + a_{n-r} V_i(\psi^{(n-r)}) = 0, \quad i = \overline{n+1, 2n}$$

or

$$V_i\left(a_1\psi^{(1)} + a_2\psi^{(2)} + \cdots + a_{n-r}\psi^{(n-r)}\right) = 0, \quad i = \overline{n+1, 2n}.$$

Thus, setting

$$\psi(t) = a_1\psi^{(1)}(t) + a_2\psi^{(2)}(t) + \cdots + a_{n-r}\psi^{(n-r)}(t), \tag{1.34}$$

we have

$$V_i(\psi) = 0 \quad (i = 1, 2, \ldots, n, n+1, \ldots, 2n);$$

in a more detailed form

$$\begin{gathered} \hat{b}^{(1)}_{m1}\psi_1(T) + \cdots + \hat{b}^{(1)}_{mn}\psi_n(T) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \hat{b}^{(1)}_{11}\psi_1(T) + \cdots + \hat{b}^{(1)}_{1n}\psi_n(T) = 0, \\ \hat{b}^{(0)}_{n-m,1}\psi_1(0) + \cdots + \hat{b}^{(0)}_{n-m,n}\psi_n(0) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \hat{b}^{(0)}_{11}\psi_1(0) + \cdots + \hat{b}^{(0)}_{1n}\psi_n(0) = 0, \\ \bar{b}^{(1)}_{n-m,1}\psi_1(T) + \cdots + \bar{b}^{(1)}_{n-m,n}\psi_n(T) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \bar{b}^{(1)}_{11}\psi_1(T) + \cdots + \bar{b}^{(1)}_{1n}\psi_n(T) = 0, \\ \bar{b}^{(0)}_{m,1}\psi_1(0) + \cdots + \bar{b}^{(0)}_{m,n}\psi_n(0) = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \bar{b}^{(0)}_{11}\psi_1(0) + \cdots + \bar{b}^{(0)}_{1n}\psi_n(0) = 0. \end{gathered} \tag{1.35}$$

Show that the determinant of the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & \cdots & 0 & \hat{b}^{(1)}_{m1} & \cdots & \hat{b}^{(1)}_{mn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \hat{b}^{(1)}_{11} & \cdots & \hat{b}^{(1)}_{1n} \\ \hat{b}^{(0)}_{n-m,1} & \cdots & \hat{b}^{(0)}_{n-m,n} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hat{b}^{(0)}_{11} & \cdots & \hat{b}^{(0)}_{1n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \bar{b}^{(1)}_{n-m,1} & \cdots & \bar{b}^{(1)}_{n-m,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \bar{b}^{(1)}_{11} & \cdots & \bar{b}^{(1)}_{1n} \\ \bar{b}^{(0)}_{m,1} & \cdots & \bar{b}^{(0)}_{m,n} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{b}^{(0)}_{11} & \cdots & \bar{b}^{(0)}_{1n} & 0 & \cdots & 0 \end{pmatrix} := G \tag{1.36}$$

of system (1.35) with respect to $\psi_1(0), \ldots, \psi_n(0), \psi_1(T), \ldots, \psi_n(T)$ is not equal to zero. By virtue of the Laplace theorem (see, e.g. [7], p. 51), the sum of all nth order minors in the last n rows of matrix G multiplied by their algebraic complements equals the matrix determinant. However, in the rows of matrix G that have numbers $n+1, \ldots, 2n$ there is only one nth order minor: its elements are in the columns with numbers $i_1, \ldots, i_m$ and rows with numbers $2n-m+1, \ldots 2n$ that form matrix $(B^T_{01})^{-1}$, and also in the columns $n+i'_1, \ldots, n+i'_m$ and rows $n+1, \ldots, 2n-m$ that form matrix $(B^T_{11})^{-1}$. Its complement equals the determinant situated in in the rows $1, \ldots, m$ and columns $n+j'_1, \ldots, n+j'_l$ forming a matrix E_m, and also in the rows $m+1, \ldots, n$ and columns $j_1, \ldots, j_l$ forming a matrix E_{n-m}. Calculating these minors with the help of the Laplace theorem, we obtain

$$\det G = (-1)^{i_1+\cdots+i_m+2n-m+1+\cdots+2n+n+i'_1+\cdots+n+i'_{n-m}+(n+1)+\cdots+2n-m} \times$$

$$\times (-1)^{i_1+\cdots+i_m+2n-m+1+\cdots+2n} \det (B^T_{01})^{-1} \det (B^T_{11})^{-1} (-1)^{n+j'_1+\cdots+n+j'_m+1+\cdots+m} =$$

$$= (-1)^{i'_1+\cdots+i'_{n-m}+j'_1+\cdots+j'_m} (-1)^{n^2} (-1)^{1+\cdots+m+(n+1)+\cdots+2n-m} \det (B^T_{01})^{-1} \det (B^T_{11})^{-1} =$$

$$= (-1)^{m^2} \det (B^T_{01})^{-1} \det (B^T_{11})^{-1} = \frac{(-1)^m}{\det B_{01} \det B_{11}} \neq 0.$$

Thus, linear equation system (1.35) has only the trivial solution $\psi_1(0) = \cdots = \psi_n(0) = \psi_1(T) = \cdots = \psi_n(T) = 0$; therefore, $\psi(t) \equiv 0$ which contradicts the linear independence of functions $\psi^{(1)}(t), \ldots, \psi^{(n-r)}(t)$ (see equality (1.34)). Finally, linear equation system (1.31) has $n-r$ linearly independent solutions (1.32) so that the rank of matrix (1.9) of system (1.31) does not exceed r which is impossible because this rank equals n according to the assumption.

We see that our initial assumption that the rank of matrix (1.29) is less than n leads to contradiction. Consequently, the rank of this matrix equals n and homogeneous BVP (1.3), (1.4) has only the trivial solution.

The reasoning above shows that the following statement is valid.

Theorem 1.2. *If homogeneous BVP (1.3), (1.4) has only the trivial solution, then the corresponding conjugate BVP (1.23), (1.24) also has only the trivial solution.*

Theorem 1.3. *Under the conditions of Theorem 2 inhomogeneous BVP (1.25), (1.26) has one and only one solution.*

Proof. According to Theorem 2, BVP (1.23), (1.24) conjugate to (1.3), (1.4) has only the trivial solution. Literally repeating the proof of Theorem 1, we obtain the required result. □

2 Statement of the minimax estimation problem and its reduction to an optimal control problem

Let a vector-function
$$y(t) = H(t)\varphi(t) + \xi(t), \tag{2.1}$$
with the values form the space R^l be observed on an interval $(\alpha,\beta) \subseteq (0,T)$; here $H(t)$ is an $l \times n$ matrix with the entries that are continuous functions on $[\alpha,\beta]$, $\xi(t)$ is a random vector process with zero expectation $M\xi(t)$ and unknown $l \times l$ correlation matrix $R(t,s) = M\xi(t)\xi^T(s)$. Let a vector-function $\varphi(t)$ be a solution to BVP (1.1), (1.2).

Denote by V the set of random vector processes $\tilde{\xi}(t)$ with zero expectation $M\tilde{\xi}(t)$ and second moments $M\tilde{\xi}(t)^2$ integrable on (α,β) such that their correlation matrix $\tilde{R}(t,s)$ belong to the space
$$\left\{\tilde{R} : \int_\alpha^\beta Sp\,[Q(t)\tilde{R}(t,t)]dt \le 1\right\}. \tag{2.2}$$

Set
$$G = \Big\{\tilde{F} := (\tilde{f}_0, \tilde{f}_1, \tilde{f}(\cdot)) : (Q_0(\tilde{f}_0 - f_0^{(0)}), \tilde{f}_0 - f_0^{(0)})_m + (Q_1(\tilde{f}_1 - f_1^{(0)}), \tilde{f}_1 - f_1^{(0)})_{n-m}$$
$$+ \int_0^T (Q_2(t)(\tilde{f}(t) - f^{(0)}), \tilde{f}(t) - f^{(0)})_n dt \le 1\Big\}, \tag{2.3}$$

where $f_0^{(0)} \in \mathbb{R}^m$, $f_1^{(0)} \in \mathbb{R}^{n-m}$ are given vectors; $f^{(0)}(t) = (f_1^{(0)}(t), f_2^{(0)}(t) \dots f_n^{(0)}(t))^T$ is a given vector-function with the components belonging to the space $L^2(0,T)$; $Q(t)$, Q_0, Q_1, and $Q_2(t)$ are positive definite matrices of dimensions $l \times l$, $m \times m$, $(n-m) \times (n-m)$, and $n \times n$, respectively, the entries of $Q(t)$, $Q^{-1}(t)$, and $Q_2(t)$, $Q_2^{-1}(t)$ are continuous on $[\alpha,\beta]$ and $[0,T]$; and $\operatorname{Sp} B = \sum_{i=1}^l b_{ii}$ denotes the trace of the matrix $B = \{b_{ij}\}_{i,j=1}^l$.

Assume that the right-hand sides $f(\cdot)$, f_0, and f_1 of equation (1.1) and boundary conditions (1.2) are not known exactly and it is known only that the element $F := (f_0, f_1, f(\cdot))$ belongs to a set G and, additionally, $\xi(t) \in V$.

We will look for an estimation of the inner product
$$(a, \varphi(s))_n, \tag{2.4}$$
in the class of estimates linear with respect to observations that have the form
$$\widehat{(a, \varphi(s))}_n = \int_\alpha^s (u_1(t), y(t))_l dt + \int_s^\beta (u_2(t), y(t))_l dt + c, \tag{2.5}$$
where $s \in (\alpha,\beta)$ and a are vectors belonging to $\mathbb{R}^n$, $u_i(t)$, $i = 1,2$ are vector-functions belonging, respectively, to $L^2(\alpha,s)$ and $L^2(s,\beta)$, and c is certain constant. Set $u := (u_1,u_2) \in H := L^2(\alpha,s) \times L^2(s,\beta) = L^2(\alpha,\beta)$.

An estimate
$$\widehat{\widehat{(a, \varphi(s))}}_n = \int_\alpha^s (\hat{u}_1(t), y(t))_l dt + \int_s^\beta (\hat{u}_2(t), y(t))_l dt + \hat{c}$$
for which vector-function $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))$ and constant $\hat{c}$ are determined from the condition
$$\sigma(u,c) := \sup_{\tilde{F}\in G, \tilde{\xi}\in V} M|(a, \tilde{\varphi}(s))_n - \widehat{(a, \tilde{\varphi}(s))}_n|^2 \to \inf_{u\in H, c\in\mathbb{R}} := \sigma^2,$$
where $\tilde{\varphi}$ is a solution to BVP (1.1), (1.2) at $f(t) = \tilde{f}(t)$, $f_0 = \tilde{f}_0$, $f_1 = \tilde{f}_1$, and
$$\widehat{(a, \tilde{\varphi}(s))}_n = \int_\alpha^s (u_1(t), \tilde{y}(t))_l dt + \int_s^\beta (u_2(t), \tilde{y}(t))_l dt + c, \quad \tilde{y}(t) = H(t)\tilde{\varphi}(t) + \tilde{\xi}(t),$$

will be called a minimax estimate of inner product $(a,\varphi(s))_n$. The quantity

$$\sigma=\{\sup_{\tilde F\in G,\tilde\xi\in V} M[(a,\tilde\varphi(s))_n-\widehat{(a,\tilde\varphi(s))_n}]^2\}^{1/2}$$

will be called the minimax estimation error.

We see that the minimax mean square estimate of inner product $(a,\varphi(s))_n$ is an estimate at which the maximum mean square estimation error calculated for the worst realization of perturbations attains its minimum.

In this section, we will show that solution to the minimax estimation problem is reduced to the solution of a certain optimal control problem.

For every fixed $u:=(u_1,u_2)\in H$ introduce vector-functions $z_1(\cdot;u)\in H^1(0,\alpha)^n$, $z_2(\cdot;u)\in H^1(\alpha,s)^n$, $z_3(\cdot;u)\in H^1(s,\beta)^n$, and $z_4(\cdot;u)\in H^1(\beta,T)^n$ as solutions to the following BVP:

$$L^*z_1(t;u)=0,\quad 0<t<\alpha,\quad \hat B_0 z_1(0;u)=0,$$

$$\begin{aligned}L^*z_2(t;u)&=-H^T(t)u_1(t),\quad \alpha<t<s,\quad z_2(\alpha;u)=z_1(\alpha;u),\\ L^*z_3(t;u)&=-H^T(t)u_2(t),\quad s<t<\beta,\quad z_3(s;u)=z_2(s;u)-a,\\ L^*z_4(t;u)&=0,\quad \beta<t<T,\quad z_4(\beta;u)=z_3(\beta;u),\quad \hat B_1 z_4(T;u)=0.\end{aligned}\tag{2.6}$$

Lemma 2.1. *Determination of the minimax estimate of inner product $(a,\varphi(s))_n$ is equivalent to the problem of optimal control of the system described by BVP (2.6) with the cost function*

$$\begin{aligned}I(u)&=(Q_0^{-1}\bar B_0 z_1(0;u),\bar B_0 z_1(0;u))_m+(Q_1^{-1}\bar B_1 z_4(T;u),\bar B_1 z_4(T;u))_{n-m}\\ &+\int_0^\alpha (Q_2^{-1}(t)z_1(t;u),z_1(t;u))_n dt+\int_\alpha^s (Q_2^{-1}(t)z_2(t;u),z_2(t;u))_n dt\\ &+\int_s^\beta (Q_2^{-1}(t)z_3(t;u),z_3(t;u))_n dt+\int_\beta^T (Q_2^{-1}(t)z_4(t;u),z_4(t;u))_n dt\\ &+\int_\alpha^s \left(Q^{-1}(t)u_1(t),u_1(t)\right)_l dt+\int_s^\beta \left(Q^{-1}(t)u_2(t),u_2(t)\right)_l dt.\end{aligned}\tag{2.7}$$

Proof. Show first that BVP (2.6) is uniquely solvable under the condition that functions $u_1(t)$ and $u_2(t)$ belong, respectively, to the spaces $L^2(\alpha,s)$ and $L^2(s,\beta)$.

Since homogeneous BVP (1.3), (1.4) has only the trivial solution, the BVP

$$L^*\psi(t)=g(t),\quad 0<t<T,\quad \hat B_0\psi(0)=0,\quad \hat B_1\psi(T)=0\tag{2.8}$$

has, in line with Theorem 3, the unique solution for any right-hand side, in particular, at

$$g(t)=g(t;u)=\begin{cases}0, & 0<t<\alpha;\\ -H^T(t)u_1(t), & \alpha<t<s;\\ -H^T(t)u_2(t), & s<t<\beta;\\ 0, & \beta<t<T.\end{cases}\tag{2.9}$$

Denote this solution by $\bar z(t;u)$ and its reductions on intervals $(0,\alpha)$, (α,s), (s,β), and (β,T) by $\bar z_1(t;u)$, $\bar z_2(t;u)$, $\bar z_3(t;u)$, and $\bar z_4(t;u)$, respectively. Note that function $\bar z(t;u)$ is absolutely continuous on $[0,T]$ (see [1]).

Let us show that the problem

$$L^*\bar{\bar z}^{(1)}(t)=0,\quad 0<t<\alpha,\quad \hat B_0\bar{\bar z}^{(1)}(0)=0,$$

$$L^*\bar{\bar{z}}^{(2)}(t) = 0, \quad \alpha < t < s, \quad \bar{\bar{z}}^{(2)}(\alpha; u) = \bar{\bar{z}}^{(1)}(\alpha; u),$$
$$L^*\bar{\bar{z}}^{(3)}(t) = 0, \quad s < t < \beta, \quad \bar{\bar{z}}^{(3)}(s; u) = \bar{\bar{z}}^{(2)}(s) - a,$$
$$L^*\bar{\bar{z}}^{(4)}(t) = 0, \quad \beta < t < T, \quad \bar{\bar{z}}^{(4)}(\beta) = \bar{\bar{z}}^{(3)}(\beta), \quad \hat{B}_1\bar{\bar{z}}^{(4)}(T) = 0 \tag{2.10}$$

has one and only one solution at any vector $a \in \mathbb{R}^n$.

Denote by $\bar{\bar{z}}_i^{(j)}(t)$, $i = \overline{1,n}$, $j = \overline{1,4}$, the coordinates of vector-function $\bar{\bar{z}}^{(j)}(t)$, $j = \overline{1,4}$. Let $y_{ik}(t)$, $i,k = \overline{1,n}$ is the fundamental system of solutions of the equation system $L^*z(t) = 0$ on $[0,T]$. The we can represent functions $\bar{\bar{z}}_i^{(j)}(t)$, $i = \overline{1,n}$, $j = \overline{1,4}$, in the form

$$\bar{\bar{z}}_i^{(j)}(t) = \sum_{k=1}^{n} c_k^{(j)} y_{ik}(t),$$

where $c_k^{(j)}$ are constants. Taking into account the boundary conditions at the points $t = 0, T$ and conjugation conditions at $t = \alpha,\ s,\ \beta$ in (2.10), we see that the solution to BVP (2.10) is equivalent to the solution of the following linear equation system with $4n$ unknowns $c_k^{(j)}$, $k = \overline{1,n}$, $j = \overline{1,4}$:

$$\sum_{k=1}^{n} a_{ik}^0 c_k^{(1)} = 0, \quad i = \overline{1, n-m}, \tag{2.11}$$

$$\sum_{k=1}^{n} y_{ik}(\alpha)(c_k^{(1)} - c_k^{(2)}) = 0, \quad i = \overline{1,n}, \tag{2.12}$$

$$\sum_{k=1}^{n} y_{ik}(s)(c_k^{(2)} - c_k^{(3)}) = a_i, \quad i = \overline{1,n}, \tag{2.13}$$

$$\sum_{k=1}^{n} y_{ik}(\beta)(c_k^{(3)} - c_k^{(4)}) = 0, \quad i = \overline{1,n}, \tag{2.14}$$

$$\sum_{k=1}^{n} a_{ik}^1 c_k^{(4)} = 0, \quad i = \overline{1,m}, \tag{2.15}$$

where

$$a_{ik}^0 = \sum_{r=1}^{n} \hat{b}_{ir}^{(0)} y_{rk}(0), \quad i = \overline{1,n-m}, \quad k = \overline{1,n},$$

$$a_{ik}^1 = \sum_{s=1}^{n} \hat{b}_{is}^{(1)} y_{sk}(T), \quad i = \overline{1,m}, \quad k = \overline{1,n},$$

a_i, $i = \overline{1,n}$, denote the coordinates of vector a, and $\hat{b}_{ir}^{(0)}$, $i = \overline{1,n-m}$, $r = \overline{1,n}$, and $\hat{b}_{is}^{(1)}$, $i = \overline{1,m}$, $s = \overline{1,n}$, denote the entries of matrices $\hat{B}_0$ and $\hat{B}_1$, respectively.

Show that system (2.11)–(2.15) is uniquely solvable at any vector $a \in \mathbb{R}^n$. To this end, note that homogeneous system (2.11)–(2.15) (at $a = 0$) has only the trivial solution.

Indeed, setting $a = 0$ in equations (2.12) and (2.13), taking into account (2.14) and the fact that $\det\{y_{ik}(\alpha)\}_{i,k=1}^n \neq 0$, $\det\{y_{ik}(s)\}_{i,k=1}^n \neq 0$, and $\det\{y_{ik}(\beta)\}_{i,k=1}^n \neq 0$ because $y_{ik}(t)$, $i,k = \overline{1,n}$ is the fundamental system of solutions of the equation system $L^*z(t) = 0$ on $[0,T]$, we obtain

$$c_k^{(1)} = c_k^{(2)} = c_k^{(3)} = c_k^{(4)} =: c_k.$$

Coefficients c_k satisfy equations (2.11) and (2.15); therefore vector-function $\psi(t)$ with the components $\psi_i(t) = \sum_{k=1}^n c_k y_{ik}(t)$, $i = \overline{1,n}$ is a solution to conjugate BVP (1.23), (1.24) which has only the trivial solution $\psi(t) \equiv 0$ on $[0,T]$. This implies $c_k = 0$, so the homogeneous linear equation system (2.11)–(2.15) (at $a = 0$) has only the trivial solution. Consequently, system (2.11)–(2.15) and therefore BVP (2.10)

which is equivalent to this system are uniquely solvable at any vector $a \in \mathbb{R}^n$. Then vector-functions $z_i(t;u) = \bar{z}_i(t;u) + \bar{\bar{z}}^{(i)}(t)$, $i = \overline{1,4}$, form the unique solution to BVP (2.6).

Show next that the determination of the minimax estimate of inner product $(a, \varphi(s))_n$ is equivalent to the problem of optimal control of the system described by BVP (2.6) with the cost function (2.7).

Using the second and third equations in (2.6) and the fact that $\tilde{\varphi}$ is a solution to BVP (1.1), (1.2) at $f(t) = \tilde{f}(t)$, $f_0 = \tilde{f}_0$, and $f_1 = \tilde{f}_1$, we easily obtain the relationships

$$-\int_\alpha^s (H^T(t)u_1(t), \tilde{\varphi}(t))_n dt = (z_2(\alpha;u), \tilde{\varphi}(\alpha))_n - (z_2(s;u), \tilde{\varphi}(s))_n + \int_\alpha^s (z_2(t;u), \tilde{f}(t))_n dt,$$

$$-\int_s^\beta (H^T(t)u_2(t), \tilde{\varphi}(t))_n dt = (z_3(s;u), \tilde{\varphi}(s))_n - (z_3(\beta;u), \tilde{\varphi}(\beta))_n + \int_s^\beta (z_3(t;u), \tilde{f}(t))_n dt.$$

Taking into account the equalities

$$z_2(\alpha;u) = z_1(\alpha;u), \quad z_2(s;u) - z_3(s;u) = a, \quad z_3(\beta;u) = z_4(\beta;u),$$

$$(z_1(\alpha;u), \tilde{\varphi}(\alpha))_n = \int_0^\alpha d(z_1(t;u), \tilde{\varphi}(t))_n + (z_1(0;u), \tilde{\varphi}(0))_n,$$

$$(z_4(\beta;u), \tilde{\varphi}(\beta))_n = -\int_\beta^T d(z_4(t;u), \tilde{\varphi}(t))_n + (z_4(T;u), \tilde{\varphi}(T))_n,$$

and that (we refer to the reasoning on p. 7)

$$(z_1(0;u), \tilde{\varphi}(0))_n = \left(\bar{B}_0 z_1(0;u), B_0\tilde{\varphi}(0)\right)_m + \left(\hat{B}_0 z_1(0;u), \tilde{B}_0\tilde{\varphi}(0)\right)_{n-m} = \left(\bar{B}_0 z_1(0;u), \tilde{f}_0\right)_m,$$

$$(z_4(T;u), \tilde{\varphi}(T))_n = \left(\bar{B}_1 z_4(T;u), B_1\tilde{\varphi}(T)\right)_{n-m} + \left(\hat{B}_1 z_4(T;u), \tilde{B}_1\tilde{\varphi}(T)\right)_m = \left(\bar{B}_1 z_4(T;u), \tilde{f}_1\right)_{n-m},$$

we obtain

$$(a, \tilde{\varphi}(s))_n - \widehat{(a, \tilde{\varphi}(s))}_n = (z_2(s;u), \tilde{\varphi}(s))_n - (z_3(s;u), \tilde{\varphi}(s))_n - \widehat{(a, \tilde{\varphi}(s))}_n =$$

$$= (z_2(\alpha;u), \tilde{\varphi}(\alpha))_n + \int_\alpha^s (H^T(t)u_1(t), \tilde{\varphi}(t))_n dt + \int_\alpha^s (z_2(t;u), \tilde{f}(t))_n dt$$

$$- (z_3(\beta;u), \tilde{\varphi}(\beta))_n + \int_s^\beta (H^T(t)u_2(t), \tilde{\varphi}(t))_n dt + \int_s^\beta (z_3(t;u), \tilde{f}(t))_n dt$$

$$- \int_\alpha^s (u_1(t), \tilde{y}(t))_l dt - \int_s^\beta (u_2(t), \tilde{y}(t))_l dt - c$$

$$= \int_0^\alpha d(z_1(t;u), \tilde{\varphi}(t))_n + (z_1(0;u), \tilde{\varphi}(0))_n + \int_\alpha^s (H^T(t)u_1(t), \tilde{\varphi}(t))_n dt$$

$$+ \int_\alpha^s (z_2(t;u), \tilde{f}(t))_n dt + \int_\beta^T d(z_4(t;u), \tilde{\varphi}(t))_n - (z_4(T;u), \tilde{\varphi}(T))_n$$

$$+ \int_s^\beta (H^T(t)u_2(t), \tilde{\varphi}(t))_n dt + \int_s^\beta (z_3(t;u), \tilde{f}(t))_n dt - \int_\alpha^s (H^T(t)u_1(t), \tilde{\varphi}(t))_n dt$$

$$- \int_\alpha^s (u_1(t), \tilde{\xi}(t))_l dt - \int_s^\beta (H^T(t)u_2(t), \tilde{\varphi}(t))_n dt - \int_s^\beta (u_2(t), \tilde{\xi}(t))_l dt - c$$

$$= \int_0^\alpha \left(\frac{dz_1(t;u)}{dt}, \tilde{\varphi}(t)\right)_n dt + \int_0^\alpha \left(\frac{d\tilde{\varphi}(t)}{dt}, z_1(t;u)\right)_n dt + (\bar{B}_0 z_1(0;u), \tilde{f}_0)_m$$

$$+ \int_\alpha^s (z_2(t;u), \tilde{f}(t))_n dt + \int_\beta^T \left(\frac{dz_4(t;u)}{dt}, \tilde{\varphi}(t)\right)_n dt + \int_\beta^T \left(\frac{d\tilde{\varphi}(t)}{dt}, z_4(t;u)\right)_n dt$$

$$-(\bar{B}_1 z_4(T;u), \tilde{f}_1)_{n-m} + \int_s^\beta (z_3(t;u), \tilde{f}(t))_n dt - \int_\alpha^s (u_1(t), \tilde{\xi}(t)_l dt - \int_s^\beta (u_2(t), \tilde{\xi}(t)_l dt - c.$$

Taking into notice that

$$\frac{dz_1(t;u)}{dt} = A^T(t) z_1(t;u) \quad \text{на} \quad (0,\alpha), \qquad \frac{dz_4(t;u)}{dt} = A^T(t) z_4(t;u) \quad \text{на} \quad (\beta, T)$$

$$\text{и} \quad \frac{d\tilde{\varphi}(t)}{dt} = \tilde{f}(t) - A(t)\tilde{\varphi}(t) \quad \text{на} \quad (0,T),$$

and therefore

$$\int_0^\alpha \left(\frac{dz_1(t;u)}{dt}, \tilde{\varphi}(t) \right)_n dt + \int_0^\alpha \left(\frac{d\tilde{\varphi}(t)}{dt}, z_1(t;u) \right)_n dt$$

$$= \int_0^\alpha (A^T(t) z_1(t;u), \tilde{\varphi}(t))_n dt + \int_0^\alpha (z_1(t;u), \tilde{f}(t) - A(t)\tilde{\varphi}(t))_n dt = \int_0^\alpha (z_1(t;u), \tilde{f}(t))_n dt,$$

$$\int_\beta^T \left(\frac{dz_4(t;u)}{dt}, \tilde{\varphi}(t) \right)_n dt + \int_\beta^T \left(\frac{d\tilde{\varphi}(t)}{dt}, z_4(t;u) \right)_n dt = \int_\beta^T (z_4(t;u), \tilde{f}(t))_n dt,$$

we use the last equality to obtain

$$(a, \tilde{\varphi}(s))_n - (\widehat{a, \tilde{\varphi}(s)})_n = \left(\bar{B}_0 z_1(0;u), \tilde{f}_0 \right)_m + \int_0^T (\tilde{z}(t;u), \tilde{f}(t))_n dt$$

$$- \left(\bar{B}_1 z_4(T;u), \tilde{f}_1 \right)_{n-m} - \int_\alpha^s (u_1(t), \tilde{\xi}(t))_l dt - \int_s^\beta (u_2(t), \tilde{\xi}(t))_l dt - c =: \eta, \tag{2.16}$$

where

$$\tilde{z}(t;u) = \begin{cases} z_1(t;u), & 0 < t < \alpha; \\ z_2(t;u), & \alpha < t < s; \\ z_3(t;u), & s < t < \beta; \\ z_4(t;u), & \beta < t < T. \end{cases}$$

Recalling that $\tilde{\xi}(t)$ is a vector process with zero expectation, we use condition (2.2) and the known relationship $D\eta = M\eta^2 - (M\eta)^2$ that couples the dispersion $D\eta := M[\eta - M\eta]^2$ of random quantity η with its expectation $M\eta$, to obtain

$$M\eta = \left(\bar{B}_0 z_1(0;u), \tilde{f}_0 \right)_m + \int_0^T (\tilde{z}(t;u), \tilde{f}(t))_n dt - \left(\bar{B}_1 z_4(T;u), \tilde{f}_1 \right)_{n-m} - c,$$

$$\eta - M\eta = -\int_\alpha^s (u_1(t), \tilde{\xi}(t))_l dt - \int_s^\beta (u_2(t), \tilde{\xi}(t))_l dt,$$

$$D\eta = M[\eta - M\eta]^2 = M \left[\int_\alpha^s (u_1(t), \tilde{\xi}(t))_l dt + \int_s^\beta (u_2(t), \tilde{\xi}(t))_l dt \right]^2,$$

$$M\eta^2 = D\eta + (M\eta)^2 = M \left[\int_\alpha^s (u_1(t), \tilde{\xi}(t))_l dt + \int_s^\beta (u_2(t), \tilde{\xi}(t))_l dt \right]^2$$

$$+ \left[\left(\bar{B}_0 z_1(0;u), \tilde{f}_0 \right)_m + \int_0^T (\tilde{z}(t;u, \tilde{f}(t))_n dt - \left(\bar{B}_1 z_4(T;u), \tilde{f}_1 \right)_{n-m} - c \right]^2,$$

which yields

$$\inf_{c \in \mathbb{R}^1} \sup_{\tilde{F} \in G, \tilde{\xi} \in V} M[(a, \tilde{\varphi}(s))_n - (\widehat{a, \tilde{\varphi}(s)})_n]^2 =$$

$$= \inf_{c\in\mathbb{R}^1} \sup_{\tilde F\in G} \left[(\bar B_0 z_1(0;u),\tilde f_0)_m + \int_0^T (\tilde z(t;u),\tilde f(t))_n dt - (\bar B_1 z_4(T;u),\tilde f_1)_{n-m} - c\right]^2$$

$$+ \sup_{\tilde\xi\in V} M\left[\int_\alpha^s (u_1(t),\tilde\xi(t))_l dt + \int_s^\beta (u_2(t),\tilde\xi(t))_l dt\right]^2. \tag{2.17}$$

In order to calculate the supremum on the right-hand side of (2.17) we apply the generalized Cauchy–Bunyakovsky inequality [6]. Let us write this inequality in the form convenient for further analysis.

Lemma. *For any $f_0^{(1)}, f_0^{(2)} \in R^m$, $f_1^{(1)}, f_1^{(2)} \in R^{n-m}$, $f_1, f_2 \in (L^2(0,T))^n$, the generalized Cauchy–Bunyakovsky inequality holds*

$$\left|(f_0^{(1)}, f_0^{(2)})_m + (f_1^{(1)}, f_1^{(2)})_{n-m} + \int_0^T (f_1(t), f_2(t))_n\, dt\right| \le$$

$$\le \left\{(Q_0^{-1} f_0^{(1)}, f_0^{(1)})_m + (Q_1^{-1} f_1^{(1)}, f_1^{(1)})_{n-m} + \int_0^T (Q_2^{-1}(t) f_1(t), f_1(t))_n\, dt\right\}^{\frac12} \times$$

$$\times \left\{(Q_0 f_0^{(2)}, f_0^{(2)})_m + (Q_1 f_1^{(2)}, f_1^{(2)})_{n-m} + \int_0^T (Q_2(t) f_2(t), f_2(t))_n\, dt\right\}^{\frac12},$$

in which the equality is attained at

$$f_0^{(2)} = \lambda Q_0^{-1} f_0^{(1)}, \quad f_1^{(2)} = \lambda Q_1^{-1} f_1^{(1)}, \quad f_2(t) = \lambda Q_2^{-1} f_1(t).$$

Setting in the generalized Cauchy–Bunyakovsky inequality

$$f_0^{(1)} = \bar B_0 z_1(0;u), \quad f_1^{(1)} = -\bar B_1 z_4(T;u), \quad f_1(t) = \tilde z(t;u),$$

$$f_0^{(2)} = \tilde f_0 - f_0^{(0)}, \quad f_1^{(2)} = \tilde f_1 - f_1^{(0)}, \quad f_2(t) = \tilde f(t) - f^{(0)}(t),$$

and denoting

$$Y := (\bar B_0 z_1(0;u), \tilde f_0 - f_0^{(0)})_m - (\bar B_1 z_4(T;u), \tilde f_1 - f_1^{(0)})_{n-m} + \int_0^T (\tilde z(t;u), \tilde f(t) - f^{(0)}(t))_n\, dt$$

we obtain, in line with (2.7), the inequality

$$|Y| \le \left\{(Q_0^{-1}\bar B_0 z_1(0;u), \bar B_0 z_1(0;u))_m + (Q_1^{-1}\bar B_1 z_4(T;u), \bar B_1 z_4(T;u))_{n-m} + \right.$$

$$\left. + \int_0^T (Q_2^{-1}(t)\tilde z(t;u), \tilde z(t;u))_n dt\right\}^{\frac12} \times \left\{(Q_0(\tilde f_0 - f_0^{(0)}), \tilde f_0 - f_0^{(0)})_m + (Q_1(\tilde f_1 - f_1^{(0)}), \tilde f_1 - f_1^{(0)})_{n-m} + \right.$$

$$\left. + \int_0^T (Q_2(t)(\tilde f(t) - f^{(0)}(t)), \tilde f(t) - f^{(0)}(t))_n dt\right\}^{\frac12} \le \left\{(Q_0^{-1}\bar B_0 z_1(0;u), \bar B_0 z_1(0;u))_m \right.$$

$$\left. + (Q_1^{-1}\bar B_1 z_4(T;u), \bar B_1 z_4(T;u))_{n-m} + \int_0^T (Q_2^{-1}(t)\tilde z(t;u), \tilde z(t;u))_n dt\right\}^{\frac12} := q,$$

where the equality is attained at

$$\tilde f_0 = \pm\frac{1}{q}\, Q_0^{-1}\bar B_0 z_1(0;u) + f_0^{(0)}, \quad \tilde f_1 = \mp\frac{1}{q}\, Q_1^{-1}\bar B_1 z_4(T;u) + f_1^{(0)}, \quad \tilde f(t) = \pm\frac{1}{q}\, Q_2(t)\tilde z(t;u) + f^{(0)}(t).$$

Thus,

$$\inf_{c\in\mathbb{R}^1}\sup_{\tilde F\in G}\left[(\bar B_0 z_1(0;u),\tilde f_0)_m+\int_0^T(\tilde z(t;u),\tilde f(t))_n dt-(\bar B_1 z_4(T;u),\tilde f_1)_{n-m}-c\right]^2$$

$$=\inf_{c\in R^1}\sup_{\tilde F\in G_0}\left[(\bar B_0 z_1(0;u),\tilde f_0-f_0^{(0)})_m-(\bar B_1 z_4(T;u),\tilde f_1-f_1^{(0)})_{n-m}+\int_0^T(\tilde z(t;u),\tilde f(t)-f^{(0)}(t))_n\,dt=\right.$$

$$\left.+(\bar B_0 z_1(0;u),f_0^{(0)})_m-(\bar B_1 z_4(T;u),f_1^{(0)})_{n-m}+\int_0^T(\tilde z(t;u),f^{(0)}(t))_n\,dt-c\right]^2$$

$$=\inf_{c\in R^1}\sup_{|Y|\le q}\left[Y+(\bar B_0 z_1(0;u),f_0^{(0)})_m-(\bar B_1 z_4(T;u),f_1^{(0)})_{n-m}+\int_0^T(\tilde z(t;u),f^{(0)}(t))_n\,dt-c\right]^2=q^2$$

$$=(Q_0^{-1}\bar B_0 z_1(0;u),\bar B_0 z_1(0;u))_m+(Q_1^{-1}\bar B_1 z_4(T;u),\bar B_1 z_4(T;u))_{n-m}$$

$$+\int_0^T(Q_2^{-1}(t)\tilde z(t;u),\tilde z(t;u))_n dt \tag{2.18}$$

at $c=(\bar B_0 z_1(0;u),f_0^{(0)})_m-(\bar B_1 z_4(T;u),f_1^{(0)})_{n-m}+\int_0^T(\tilde z(t;u),f^{(0)}(t))_n\,dt$.

Calculate the second term on the right-hand side of (2.17). Setting

$$\tilde u(t)=\begin{cases}u_1(t), & \alpha<t<s,\\ u_2(t), & s<t<\beta,\end{cases}$$

and applying the generalized Cauchy$-$Bunyakovsky inequality, we have

$$M\left[\int_\alpha^s(u_1(t),\tilde\xi(t))_l dt+\int_s^\beta(u_2(t),\tilde\xi(t))_l dt\right]^2=M\left[\int_\alpha^\beta(\tilde u(t),\tilde\xi(t))_l dt\right]^2$$

$$\le M\left[\int_\alpha^\beta(Q^{-1}(t)\tilde u(t),\tilde u(t))_l dt\cdot\int_\alpha^\beta(Q(t)\xi(t),\xi(t))_l dt\right]$$

$$=\int_\alpha^\beta(Q^{-1}(t)\tilde u(t),\tilde u(t))_l dt\cdot\int_\alpha^\beta M(Q(t)\tilde\xi(t),\tilde\xi(t))_l dt. \tag{2.19}$$

Here M can be placed under the integral sign according to the Fubini theorem because we assume that $\tilde\xi(t)$ is a random process of the integrable second moment. Transform the last factor on the right-hand side of (2.19):

$$\int_\alpha^\beta M(Q(t)\tilde\xi(t),\tilde\xi(t))_l dt=\int_\alpha^\beta M(\sum_{i=1}^l(Q(t)\tilde\xi(t))_i\tilde\xi_i(t))dt$$

$$=\int_\alpha^\beta M(\sum_{i=1}^l\sum_{k=1}^l q_{ik}\tilde\xi_k(t)\tilde\xi_i(t))dt=\int_\alpha^\beta\sum_{i=1}^l\sum_{k=1}^l q_{ik}M(\tilde\xi_k(t)\tilde\xi_i(t))dt$$

$$=\int_\alpha^\beta Sp\,[Q(t)\tilde R(t,t)]dt.$$

Taking into account that (2.2) holds, we see that (2.19) yields

$$\sup_{\tilde\xi\in V}M\left[\int_\alpha^s(u_1(t),\tilde\xi(t))_l dt+\int_s^\beta(u_2(t),\tilde\xi(t))_l dt\right]^2\le$$

$$\le\int_\alpha^s\left(Q^{-1}(t)u_1(t),u_1(t)\right)_l dt+\int_s^\beta\left(Q^{-1}(t)u_2(t),u_2(t)\right)_l dt. \tag{2.20}$$

It is not difficult to check that here, the equality sign is attained at the element

$$\tilde{\xi}(t) = \begin{cases} \dfrac{\eta Q^{-1}(t)u_1(t)}{\left[\int_\alpha^s \left(Q^{-1}(t)u_1(t), u_1(t)\right)_l dt + \int_s^\beta \left(Q^{-1}(t)u_2(t), u_2(t)\right)_l dt\right]^{1/2}}, & \alpha \le t < s; \\ \dfrac{\eta Q^{-1}(t)u_2(t)}{\left[\int_\alpha^s \left(Q^{-1}(t)u_1(t), u_1(t)\right)_l dt + \int_s^\beta \left(Q^{-1}(t)u_2(t), u_2(t)\right)_l dt\right]^{1/2}}, & s < t \le \beta, \end{cases}$$

where η is a random quantity such that $M\eta = 0$ and $M\eta^2 = 1$. We conclude that statement of the lemma follows now from (2.17), (2.18), and (2.20). □

3 Representations for minimax estimates of functionals of solutions to two-point boundary value problems and estimation errors

In this section we prove the theorem concerning general form of minimax mean square estimates. Solving optimal control problem (2.6), (2.7), we arrive at the following result.

Theorem 3.1. *The minimax estimate of expression $(a, \varphi(s))$ has the form*

$$\widehat{\widehat{(a, \varphi(s))}} = \int_\alpha^s (\hat{u}_1(t), y(t))_l dt + \int_s^\beta (\hat{u}_2(t), y(t))_l dt$$

where

$$\hat{u}_1(t) = Q(t)H(t)p_2(t), \quad \hat{u}_2(t) = Q(t)H(t)p_3(t), \tag{3.1}$$

$$\hat{c} = (\bar{B}_0 z_1(0), f_0^{(0)})_m - (\bar{B}_1 z_4(T), f_1^{(0)})_{n-m} + \int_0^T (\tilde{z}(t), f^{(0)}(t))_n \, dt,$$

$$\tilde{z}(t) = \begin{cases} z_1(t), & 0 < t < \alpha; \\ z_2(t), & \alpha < t < s; \\ z_3(t), & s < t < \beta; \\ z_4(t), & \beta < t < T, \end{cases}$$

and vector-functions $p_i(t)$ and $z_i(t)$, $i = \overline{1,4}$, are determined from the solution to the equation systems

$$L^* z_1(t) = 0, \quad 0 < t < \alpha, \quad \hat{B}_0 z_1(0) = 0,$$

$$\begin{gathered} L^* z_2(t) = -H^T(t)Q(t)H(t)p_2(t), \quad \alpha < t < s, \quad z_2(\alpha) = z_1(\alpha), \\ L^* z_3(t) = -H^T(t)Q(t)H(t)p_3(t), \quad s < t < \beta, \quad z_3(s) = z_2(s) - a, \\ L^* z_4(t) = 0, \quad \beta < t < T, \quad z_4(\beta) = z_3(\beta), \quad \hat{B}_1 z_4(T) = 0, \\ L p_1(t) = Q_2^{-1}(t) z_1(t), \quad 0 < t < \alpha, \quad B_0 p_1(0) = Q_0^{-1} \bar{B}_0 z_1(0), \\ L p_2(t) = Q_2^{-1}(t) z_2(t), \quad \alpha < t < s, \quad p_2(\alpha) = p_1(\alpha), \\ L p_3(t) = Q_2^{-1}(t) z_3(t), \quad s < t < \beta, \quad p_3(s) = p_2(s), \\ L p_4(t) = Q_2^{-1}(t) z_4(t), \quad \beta < t < T, \quad p_4(\beta) = p_3(\beta), \quad B_1 p_4(T) = -Q_1^{-1} \bar{B}_1 z_4(T). \end{gathered} \tag{3.2}$$

Here $z_1, p_1 \in H^1(0,\alpha)^n$, $z_2, p_2 \in H^1(\alpha, s)^n$, $z_3, p_3 \in H^1(s,\beta)^n$, and $z_4, p_4 \in H^1(\beta, T)^n$. The minimax estimation error

$$\sigma = (a, p_2(s))_n^{1/2}. \tag{3.3}$$

System (3.2) is uniquely solvable.

Proof. We will solve optimal control problem (2.6), (2.7). Represent solutions $z_i(x;u)$, $i=\overline{1,4}$, of problem (2.6) as $z_i(t;u)=\bar z_i(t;u)+\bar{\bar z}_i(t)$, where $\bar z_1(t;u)$, $\bar z_2(t;u)$, $\bar z_3(t;u)$, $\bar z_4(t;u)$ and $\bar{\bar z}_1(t)$, $\bar{\bar z}_2(t)$, $\bar{\bar z}_3(t)$, $\bar{\bar z}_4(t)$ denote the solutions to this problem at $a=0$ and $u_1\equiv 0$, $u_2\equiv 0$, respectively. Then function (2.7) can be represented in the form

$$I(u)=\tilde I(u)+2L(u)+A,$$

where

$$\begin{aligned}\tilde I(u)&=(Q_0^{-1}\bar B_0\bar z_1(0;u),\bar B_0\bar z_1(0;u))_m+(Q_1^{-1}\bar B_1\bar z_4(T;u),\bar B_1\bar z_4(T;u))_{n-m}\\
&+\int_0^\alpha (Q_2^{-1}(t)\bar z_1(t;u),\bar z_1(t;u))_n dt+\int_\alpha^s (Q_2^{-1}(t)\bar z_2(t;u),\bar z_2(t;u))_n dt\\
&+\int_s^\beta (Q_2^{-1}(t)\bar z_3(t;u),\bar z_3(t;u))_n dt+\int_\beta^T (Q_2^{-1}(t)\bar z_4(t;u),\bar z_4(t;u))_n dt\\
&+\int_\alpha^s \left(Q^{-1}(t)u_1(t),u_1(t)\right)_l dt+\int_s^\beta \left(Q^{-1}(t)u_2(t),u_2(t)\right)_l dt,\\
L(u)&=(Q_0^{-1}\bar B_0\bar z_1(0;u),\bar B_0\bar{\bar z}_1(0))_m+(Q_1^{-1}\bar B_1\bar z_4(T;u),\bar B_1\bar{\bar z}_4(T))_{n-m}\\
&+\int_0^\alpha (Q_2^{-1}(t)\bar z_1(t;u),\bar{\bar z}_1(t))_n dt+\int_\alpha^s (Q_2^{-1}(t)\bar z_2(t;u),\bar{\bar z}_2(t))_n dt\\
&+\int_s^\beta (Q_2^{-1}(t)\bar z_3(t;u),\bar{\bar z}_3(t))_n dt+\int_\beta^T (Q_2^{-1}(t)\bar z_4(t;u),\bar{\bar z}_4(t))_n dt,\\
A&=(Q_0^{-1}\bar B_0\bar{\bar z}_1(0),\bar B_0\bar{\bar z}_1(0))_m+(Q_1^{-1}\bar B_1\bar{\bar z}_4(T),\bar B_1\bar{\bar z}_4(T))_{n-m}\\
&+\int_0^\alpha (Q_2^{-1}(t)\bar{\bar z}_1(t),\bar{\bar z}_1(t))_n dt+\int_\alpha^s (Q_2^{-1}(t)\bar{\bar z}_2(t),\bar{\bar z}_2(t))_n dt\\
&+\int_s^\beta (Q_2^{-1}(t)\bar{\bar z}_3(t),\bar{\bar z}_3(t))_n dt+\int_\beta^T (Q_2^{-1}(t)\bar{\bar z}_4(t),\bar{\bar z}_4(t))_n dt.\end{aligned}$$

Since solution $\bar z(t;u)$ of BVP (2.8) is continuous[1] with respect to right-hand side $g(t;u)$ defined by (2.9), the function $u\to\bar z(\cdot;u)$ is a linear bounded operator mapping the space $H=L^2(\alpha,s)\times L^2(s,\beta)$ to

$$H_1:=H^1(0,\alpha)\times H^1(\alpha,s)\times H^1(s,\beta)\times H^1(\beta,T).$$

Thus, $\tilde I(u)$ is a continuous quadratic form corresponding to a symmetric continuous bilinear form

$$\begin{aligned}\pi(u,v)&:=(Q_0^{-1}\bar B_0\bar z_1(0;u),\bar B_0\bar z_1(0;v))_m+(Q_1^{-1}\bar B_1\bar z_4(T;u),\bar B_1\bar z_4(T;v))_{n-m}\\
&+\int_0^\alpha (Q_2^{-1}(t)\bar z_1(t;u),\bar z_1(t;v))_n dt+\int_\alpha^s (Q_2^{-1}(t)\bar z_2(t;u),\bar z_2(t;v))_n dt\\
&+\int_s^\beta (Q_2^{-1}(t)\bar z_3(t;u),\bar z_3(t;v))_n dt+\int_\beta^T (Q_2^{-1}(t)\bar z_4(t;u),\bar z_4(t;v))_n dt\\
&+\int_\alpha^s \left(Q^{-1}(t)u_1(t),v_1(t)\right)_l dt+\int_s^\beta \left(Q^{-1}(t)u_2(t),v_2(t)\right)_l dt,\end{aligned}$$

[1]This continuous dependence follows from the representation of function $\bar z(t;u)$ in terms of Green's matrix $G^*(t,\xi)$ of BVP (2.8) (see [4], p. 115):

$$\bar z(t;u)=-\int_\alpha^s G^*(t,\xi)H^T(\xi)u_1(\xi)\,d\xi-\int_s^\beta G^*(t,\xi)H^T(\xi)u_2(\xi)\,d\xi.$$

$L(u)$ is a linear continuous functional defined on H, and A is a constant independent of u. We have

$$\tilde{I}(u) = \tilde{I}(u_1, u_2) \geq \int_\alpha^s \left(Q^{-1}(t)u_1(t), u_1(t)\right)_l dt + \int_s^\beta \left(Q^{-1}(t)u_2(t), u_2(t)\right)_l dt \geq c\|u\|_H^2, \quad \text{c=const;}$$

using Theorem 1.1 from [3], we conclude that there is one and only one element $\hat{u} = (\hat{u}_1, \hat{u}_2) \in H$ such that

$$I(\hat{u}) = I(\hat{u}_1, \hat{u}_2) = \inf_{(u_1,u_2)\in H} I(u_1, u_2) = \inf_{u_1 \in L^2(\alpha,s), u_2 \in L^2(s,\beta)} I(u_1, u_2).$$

Therefore

$$\frac{d}{d\tau} I(\hat{u}_1 + \tau v_1, \hat{u}_2 + \tau v_2) \mid_{\tau=0} = 0, \quad \forall v = (v_1, v_2) \in H.$$

Taking into consideration the latter equality, (2.7), and designations on p. 20, we obtain

$$0 = \frac{1}{2} \frac{dI(\hat{u} + \tau v)}{d\tau} \mid_{\tau=0} =$$

$$\begin{aligned}
&= \int_\alpha^s (Q^{-1}(t)\hat{u}_1, v_1)_l dt + \int_s^\beta (Q^{-1}(t)\hat{u}_2, v_2)_l dt + \\
&+ (Q_0^{-1}\bar{B}_0 z_1(0;\hat{u}), \bar{B}_0 \bar{z}_1(0;v))_m + (Q_1^{-1}\bar{B}_1 z_4(T;\hat{u}), \bar{B}_1 \bar{z}_4(T;v))_{n-m} \\
&+ \int_0^\alpha (Q_2^{-1}(t) z_1(t;\hat{u}), \bar{z}_1(t;v))_n dt + \int_\alpha^s (Q_2^{-1}(t) z_2(t;\hat{u}), \bar{z}_2(t;v))_n dt \\
&+ \int_s^\beta (Q_2^{-1}(t) z_3(t;\hat{u}), \bar{z}_3(t;v))_n dt + \int_\beta^T (Q_2^{-1}(t) z_4(t;\hat{u}), \bar{z}_4(t;v))_n dt
\end{aligned} \tag{3.4}$$

Introduce functions $p_1 \in H^1(0,\alpha)^n$, $p_2 \in H^1(\alpha,s)^n$, $p_3 \in H^1(s,\beta)^n$, and $p_4 \in H^1(\beta,T)^n$ as unique solutions to the following problem:

$$\begin{gathered}
Lp_1(t) = Q_2^{-1}(t) z_1(t;\hat{u}), \quad 0 < t < \alpha, \quad B_0 p_1(0) = Q_0^{-1}\bar{B}_0 z_1(0;\hat{u}), \\
Lp_2(t) = Q_2^{-1}(t) z_2(t;\hat{u}), \quad \alpha < t < s, \quad p_2(\alpha) = p_1(\alpha), \\
Lp_3(t) = Q_2^{-1}(t) z_3(t;\hat{u}), \quad s < t < \beta, \quad p_3(s) = p_2(s), \\
Lp_4(t) = Q_2^{-1}(t) z_4(t;\hat{u}), \quad \beta < t < T, \\
p_4(\beta) = p_3(\beta), \quad B_1 p_4(T) = -Q_1^{-1}\bar{B}_1 z_4(T;\hat{u}).
\end{gathered} \tag{3.5}$$

Now transform the sum of the last for terms on the right-hand side of (3.4) taking into notice that $(\bar{z}_1(0;v), p_1(0))_n = (\bar{B}_0 \bar{z}_1(0;v), B_0 p_1(0))_m$ and $(\bar{z}_4(T;v), p_4(T))_n = (\bar{B}_1 \bar{z}_4(T;v), B_1 p_4(T))_{n-m}$. We have

$$\begin{aligned}
&\int_0^\alpha (Q_2^{-1}(t) z_1(t;\hat{u}), \bar{z}_1(t;v))_n dt + \int_\alpha^s (Q_2^{-1}(t) z_2(t;\hat{u}), \bar{z}_2(t;v))_n dt \\
&+ \int_s^\beta (Q_2^{-1}(t) z_3(t;\hat{u}), \bar{z}_3(t;v))_n dt + \int_\beta^T (Q_2^{-1}(t) z_4(t;\hat{u}), \bar{z}_4(t;v))_n dt \\
&= \int_0^\alpha (Lp_1(t), \bar{z}_1(t;v))_n dt + \int_\alpha^s (Lp_2(t), \bar{z}_2(t;v))_n dt \\
&+ \int_s^\beta (Lp_3(t), \bar{z}_3(t;v))_n dt + \int_\beta^T (Lp_4(t), \bar{z}_4(t;v))_n dt \\
&= \int_0^\alpha (p_1(t), L^* \bar{z}_1(t;v))_n dt + (\bar{z}_1(\alpha;v), p_1(\alpha))_n - (\bar{z}_1(0;v), p_1(0))_n
\end{aligned}$$

$$+\int_\alpha^s (p_2(t), L^*\bar z_2(t;v))_n dt + (\bar z_2(s;v), p_2(s))_n - (\bar z_2(\alpha;v), p_2(\alpha))_n$$
$$+\int_s^\beta (p_3(t), L^*\bar z_3(t;v))_n dt + (\bar z_3(\beta;v), p_3(\beta))_n - (\bar z_3(s;v), p_3(s))_n$$
$$+\int_\beta^T (p_4(t), L^*\bar z_4(t;v))_n dt + (\bar z_4(T;v), p_4(T))_n - (\bar z_4(\beta;v), p_4(\beta))_n$$
$$= -(\bar z_1(0;v), p_1(0))_n + (\bar z_4(T;v), p_4(T))_n$$
$$-\int_\alpha^s (p_2(t), H^T(t)v_1(t))_n dt - \int_s^\beta (p_3(t), H^T(t)v_2(t))_n dt + (\bar z_2(s;v) - \bar z_3(s;v), p_2(s))_n$$
$$= -(\bar B_0 \bar z_1(0;v), B_0 p_1(0))_m + (\bar B_1 \bar z_4(T;v), B_1 p_4(T))_{n-m}$$
$$-\int_\alpha^s (H(t)p_2(t), v_1(t))_l dt - \int_s^\beta (H(t)p_3(t), v_2(t))_l dt$$
$$= -(\bar B_0 \bar z_1(0;v), Q_0^{-1}\bar B_0 z_1(0;\hat u))_m - (B_1 \bar z_4(T;v), Q_1^{-1}\bar B_1 z_4(T;\hat u))_{n-m}$$
$$-\int_\alpha^s (H(t)p_2(t), v_1(t))_l dt - \int_s^\beta (H(t)p_3(t), v_2(t))_l dt \tag{3.6}$$

From equalities (3.4)–(3.6) it follows that

$$\int_\alpha^s (Q^{-1}(t)\hat u_1(t), v_1(t))_l dt + \int_s^\beta (Q^{-1}(t)\hat u_2(t), v_2(t))_l dt$$
$$= \int_\alpha^s (H(t)p_2(t), v_1(t))_l dt + \int_s^\beta (H(t)p_3(t), v_2(t))_l dt,$$

so that

$$Q^{-1}(t)\hat u_1(t) = H(t)p_2(t), \quad Q^{-1}(t)\hat u_2(t) = H(t)p_3(t),$$
$$\hat u_1(t) = Q(t)H(t)p_2(t), \quad \hat u_2(t) = Q(t)H(t)p_3(t). \tag{3.7}$$

Functions $p_1(t)$, $p_2(t)$, $p_3(t)$, and $p_4(t)$ are absolutely continuous on segments $[0,\alpha]$, $[\alpha,s]$, $[s,\beta]$, and $[\beta,T]$, respectively, as solutions to BVP (3.5); therefore, functions $\hat u_1(t)$ and $\hat u_2(t)$ that perform optimal control are continuous on $[\alpha,s]$ and $[s,\beta]$. Replacing in (2.6) functions $u_1(t)$ and $u_2(t)$ by $\hat u_1(t)$ and $\hat u_2(t)$ defined by formulas (3.7) and denoting $z(t) = z(t;\hat u)$ we arrive at problem (3.2) and equalities (3.1).

Taking into consideration the way this problem was formulated we can state that its unique solvability follows from the fact that functional (2.7) has one minimum point $\hat u$.

Now let us prove representation (3.3). Substituting into formula $\sigma^2 = I(\hat u)$ expressions (3.1) for $\hat u_1(t)$ and $\hat u_2(t)$, we have

$$\sigma^2 = (Q_0^{-1}\bar B_0 z_1(0), \bar B_0 z_1(0))_m + (Q_1^{-1}\bar B_1 z_4(T), \bar B_1 z_4(T))_{n-m}$$
$$+\int_0^\alpha (Q_2^{-1}(t)z_1(t), z_1(t))_n dt + \int_\alpha^s (Q_2^{-1}(t)z_2(t), z_2(t))_n dt$$
$$+\int_s^\beta (Q_2^{-1}(t)z_3(t), z_3(t))_n dt + \int_\beta^T (Q_2^{-1}(t)z_4(t), z_4(t))_n dt$$
$$+\int_\alpha^s (Q(t)H(t)p_2(t), H(t)p_2(t))_l\, dt + \int_s^\beta (Q(t)H(t)p_3(t), H(t)p_3(t))_l\, dt. \tag{3.8}$$

Next, we can apply the reasoning similar to that on p. 7 and use (3.2) to obtain

$$(z_1(0), p_1(0))_n = (\bar B_0 z_1(0), B_0 p_1(0))_m = (\bar B_0 z_1(0), Q_0^{-1}\bar B_0 z_1(0))_m,$$

which yields

$$\int_0^\alpha (Q_2^{-1}(t)z_1(t), z_1(t))_n dt + \int_\alpha^s (Q(t)H(t)p_2(t), H(t)p_2(t))_l \, dt$$

$$= \int_0^\alpha (Lp_1(t), z_1(t))_n dt - \int_\alpha^s (L^* z_2(t), p_2(t))_n dt$$

$$= \int_0^\alpha (p_1(t), L^* z_1(t))_n dt + (z_1(\alpha), p_1(\alpha))_n - (z_1(0), p_1(0))_n$$

$$- \int_\alpha^s (z_2(t), Lp_2(t))_n dt - (z_2(\alpha), p_2(\alpha))_n + (z_2(s), p_2(s))_n$$

$$= -(z_1(0), p_1(0))_n - \int_\alpha^s (z_2(t), Q_2^{-1}(t) z_2(t))_n dt + (z_2(s), p_2(s))_n$$

$$= - \int_\alpha^s (Q_2^{-1}(t)z_2(t), z_2(t))_n dt - (\bar{B}_0 z_1(0), Q_0^{-1} \bar{B}_0 z_1(0))_m + (z_2(s), p_2(s))_n \tag{3.9}$$

In a similar manner, using the equality

$$(z_4(T), p_4(T))_n = (\bar{B}_1 z_4(T), B_1 p_4(T))_{n-m} = -(\bar{B}_1 z_4(T), Q_1^{-1} \bar{B}_1 z_4(T))_{n-m},$$

we obtain

$$\int_\beta^T (Q_2^{-1}(t)z_4(t), z_4(t))_n dt + \int_s^\beta (Q(t)H(t)p_3(t), H(t)p_3(t))_l \, dt$$

$$= \int_\beta^T (Lp_4(t), z_4(t))_n dt - \int_s^\beta (L^* z_3(t), p_3(t))_n dt$$

$$= \int_\beta^T (p_4(t), L^* z_4(t))_n dt + (p_4(T), z_4(T))_n - (p_4(\beta), z_4(\beta))_n$$

$$- \int_s^\beta (z_3(t), Lp_3(t))_n dt + (z_3(\beta), p_3(\beta))_n - (z_3(s), p_3(s))_n$$

$$= (p_4(T), z_4(T))_n - \int_s^\beta (z_3(t), Q_2^{-1}(t) z_3(t))_n dt - (z_3(s), p_2(s))_n$$

$$= - \int_s^\beta (Q_2^{-1}(t)z_3(t), z_3(t))_n dt - (\bar{B}_1 z_4(T), Q_1^{-1} \bar{B}_1 z_4(T))_{n-m} - (z_3(s), p_2(s))_n. \tag{3.10}$$

Relationships (3.8)–(3.10) yield

$$\sigma^2 = (a, p_2(s)),$$

which is to be proved. □

Obtain now another representation for the minimax estimate of quantity $(a, \varphi(s))_n$ which is independent of vector a. To this end, introduce vector-functions $\hat{p}_1, \hat{\varphi}_1 \in H^1(0,\alpha)^n$, $\hat{p}_2, \hat{\varphi}_2 \in H^1(\alpha, s)^n$, $\hat{p}_3, \hat{\varphi}_3 \in H^1(s,\beta)^n$, and $\hat{p}_4, \hat{\varphi}_4 \in H^1(\beta, T)^n$ as solutions to the equation system

$$L^* \hat{p}_1(t) = 0, \quad 0 < t < \alpha, \quad \hat{B}_0 \hat{p}_1(0) = 0,$$

$$L^* \hat{p}_2(t) = H^T(t)Q(t)(y(t) - H(t)\hat{\varphi}_2(t)), \quad \alpha < t < s, \quad \hat{p}_2(\alpha) = \hat{p}_1(\alpha),$$
$$L^* \hat{p}_3(t) = H^T(t)Q(t)(y(t) - H(t)\hat{\varphi}_3(t)), \quad s < t < \beta, \quad \hat{p}_3(s) = \hat{p}_2(s),$$
$$L^* \hat{p}_4(t) = 0, \quad \beta < t < T, \quad \hat{p}_4(\beta) = \hat{p}_3(\beta), \quad \hat{B}_1 \hat{p}_4(T) = 0, \tag{3.11}$$
$$L\hat{\varphi}_1(t) = Q_2^{-1}(t)\hat{p}_1(t) + f^{(0)}(t), \quad 0 < t < \alpha, \quad B_0 \hat{\varphi}_1(0) = Q_0^{-1} \bar{B}_0 \hat{p}_1(0) + f_0^{(0)},$$
$$L\hat{\varphi}_2(t) = Q_2^{-1}(t)\hat{p}_2(t) + f^{(0)}(t), \quad \alpha < t < s, \quad \hat{\varphi}_2(\alpha) = \hat{\varphi}_1(\alpha),$$

$$L\hat{\varphi}_3(t) = Q_2^{-1}(t)\hat{p}_3(t) + f^{(0)}(t), \quad s < t < \beta, \quad \hat{\varphi}_3(s) = \hat{\varphi}_2(s),$$
$$L\hat{\varphi}_4(t) = Q_2^{-1}(t)\hat{p}_4(t) + f^{(0)}(t), \quad \beta < t < T,$$
$$\hat{\varphi}_4(\beta) = \hat{\varphi}_3(\beta), \quad B_1\hat{\varphi}_4(T) = -Q_1^{-1}\bar{B}_1\hat{p}_4(T) - f_1^{(0)}$$

at realizations y that belong with probability 1 to space $L^2(\alpha, \beta)$.

Note that unique solvability of problem (3.11) can be proved similarly to the case of (3.2). Namely, one can show that solutions to the problem of optimal control of the system

$$L^*\hat{p}_1(t; v) = 0, \quad 0 < t < \alpha, \quad \hat{B}_0\hat{p}_1(0; v) = 0,$$

$$L^*\hat{p}_2(t; v) = d(t) - H^T(t)v_1(t), \quad \alpha < t < s, \quad \hat{p}_1(\alpha; v) = \hat{p}_2(\alpha; v),$$
$$L^*\hat{p}_3(t; v) = d(t) - H^T(t)v_2(t), \quad s < t < \beta, \quad \hat{p}_2(s; v) = \hat{p}_3(s; v),$$
$$L^*\hat{p}_4(t; v) = 0, \quad \beta < t < T, \quad \hat{p}_4(\beta; v) = \hat{p}_3(\beta; v), \quad \hat{B}_1\hat{p}_4(T; v) = 0$$

with the cost function

$$J(v) = \left(Q_0^{-1}(\bar{B}_0\hat{p}_1(0; v) + Q_0 f_0^{(0)}), \bar{B}_0\hat{p}_1(0; v) + Q_0 f_0^{(0)}\right)_m$$
$$+ \left(Q_1^{-1}(\bar{B}_1\hat{p}_4(T; v) + Q_1 f_1^{(0)}), \bar{B}_1\hat{p}_4(T; v) + Q_1 f_1^{(0)}\right)_{n-m}$$
$$+ \int_0^\alpha (Q_2^{-1}(t)(\hat{p}_1(t; v) + Q_2(t)f^{(0)}(t)), \hat{p}_1(t; v) + Q_2(t)f^{(0)}(t))_n dt$$
$$+ \int_\alpha^s (Q_2^{-1}(t)(\hat{p}_2(t; v) + Q_2(t)f^{(0)}(t)), \hat{p}_2(t; v) + Q_2(t)f^{(0)}(t))_n dt$$
$$+ \int_s^\beta (Q_2^{-1}(t)(\hat{p}_3(t; v) + Q_2(t)f^{(0)}(t)), \hat{p}_3(t; v) + Q_2(t)f^{(0)}(t))_n dt$$
$$+ \int_\beta^T (Q_2^{-1}(t)(\hat{p}_4(t; v) + Q_2(t)f^{(0)}(t)), \hat{p}_4(t; v) + Q_2(t)f^{(0)}(t))_n dt$$
$$+ \int_\alpha^s \left(Q^{-1}(t)v_1(t), v_1(t)\right)_l dt + \int_s^\beta \left(Q^{-1}(t)v_2(t), v_2(t)\right)_l dt \to \min_{v=(v_1,v_2)\in H},$$
$$d(t) = H^T(t)Q(t)y(t), \quad \alpha < t < \beta,$$

can be reduced to the solution of problem (3.11) where the optimal control $\hat{v} = (\hat{v}_1, \hat{v}_2)$ is expressed in terms of the solution to this problem as $\hat{v}_1 = Q(t)H(t)\hat{\varphi}_2(t)$, $\hat{v}_2 = Q(t)H(t)\hat{\varphi}_3(t)$; the unique solvability of the problem follows from the existence of the unique minimum point $\hat{v}$ of functional $J(v)$.

Considering system (3.11) at realizations y it is easy to see that its solution is continuous with respect to the right-hand side. This property enables us to conclude, using the general theory of linear continuous transformations of random processes, that these solutions, i.e. the functions $\hat{p}_i(t) = \hat{p}_i(t, \omega)$, $\hat{\varphi}_1(t) = \hat{\varphi}_1(t, \omega)$, $i = \overline{1, 4}$, considered as random fields have finite second moments.

Theorem 3.2. *The following representation is valid*

$$\widehat{(a, \varphi(s))_n} = (a, \hat{\varphi}_2(s))_n.$$

Proof. By virtue of (3.1) and (3.11),

$$\widehat{(a,\varphi(s))}_n = \int_\alpha^s (\hat u_1(t), y(t))_l dt + \int_s^\beta (\hat u_2(t), y(t))_l dt + \hat c$$

$$= \int_\alpha^s (Q(t)H(t)p_2(t), y(t))_l dt + \int_\alpha^s (Q(t)H(t)p_3(t), y(t))_l dt + \hat c$$

$$= \int_\alpha^s (p_2(t), H^T(t)Q(t)y(t))_n dt + \int_\alpha^s (p_3(t), H^T(t)Q(t)y(t))_n dt + \hat c. \tag{3.12}$$

Next,

$$\int_\alpha^s (p_2(t), H^T(t)Q(t)y(t))_l dt =$$

$$= \int_\alpha^s (p_2(t), L^*\hat p_2(t))_n dt + \int_\alpha^s (p_2(t), H^T(t)Q(t)H(t)\hat\varphi_2(t))_n dt$$

$$= \int_\alpha^s (Lp_2(t), \hat p_2(t))_n dt + (p_2(\alpha), \hat p_2(\alpha))_n - (p_2(s), \hat p_2(s))_n$$

$$+ \int_\alpha^s (H^T(t)Q(t)H(t)p_2(t), \hat\varphi_2(t))_n dt$$

$$= \int_\alpha^s (Q_2^{-1}(t)z_2(t), \hat p_2(t))_n dt + (p_2(\alpha), \hat p_2(\alpha))_n - (p_2(s), \hat p_2(s))_n$$

$$- \int_\alpha^s (L^* z_2(t), \hat\varphi_2(t))_n dt$$

$$= \int_\alpha^s (z_2(t), L\hat\varphi_2(t))_n dt + (p_2(\alpha), \hat p_2(\alpha))_n - (p_2(s), \hat p_2(s))_n - \int_\alpha^s (z_2(t), L\hat\varphi_2(t))_n dt$$

$$+ (z_2(s), \hat\varphi_2(s))_n - (z_2(\alpha), \hat\varphi_2(\alpha))_n - \int_\alpha^s (z_2(t), f^{(0)}(t))_n dt$$

$$= (p_2(\alpha), \hat p_2(\alpha))_n - (p_2(s), \hat p_2(s))_n + (z_2(s), \hat\varphi_2(s))_n - (z_2(\alpha), \hat\varphi_2(\alpha))_n - \int_\alpha^s (z_2(t), f^{(0)}(t))_n dt. \tag{3.13}$$

Similarly,

$$\int_s^\beta (p_3(t), H^T(t)Q(t)y(t))_l dt =$$

$$= (p_3(s), \hat p_3(s))_n - (p_3(\beta), \hat p_3(\beta))_n + (z_3(\beta), \hat\varphi_3(\beta))_n - (z_3(s), \hat\varphi_3(s))_n - \int_s^\beta (z_3(t), f^{(0)}(t))_n dt \tag{3.14}$$

From (3.12), (3.13), and (3.14) it follows

$$\widehat{(a,\varphi(s))} = (p_1(\alpha), \hat p_1(\alpha))_n - \int_\alpha^s (z_2(t), f^{(0)}(t))_n dt - \int_s^\beta (z_3(t), f^{(0)}(t))_n dt$$

$$- (z_1(\alpha), \hat\varphi_1(\alpha))_n - (p_4(\beta), \hat p_4(\beta))_n + (z_4(\beta), \hat\varphi_4(\beta))_n + (a, \hat\varphi_2(s))_n + \hat c. \tag{3.15}$$

However,

$$0 = \int_0^\alpha (\hat\varphi_1(t), L^*\hat z_1(t))_n dt = \int_0^\alpha (L\hat\varphi_1(t), z_1(t))_n dt$$

$$-(z_1(\alpha), \hat\varphi_1(\alpha))_n + (z_1(0), \hat\varphi_1(0))_n$$

$$= \int_0^\alpha (z_1(t), Q_2(t)\hat p_1(t))_n dt + \int_0^\alpha (z_1(t), f^{(0)}(t))_n dt - (z_1(\alpha), \hat\varphi_1(\alpha))_n + (z_1(0), \hat\varphi_1(0))_n, \tag{3.16}$$

$$0 = \int_0^\alpha (L^*\hat{p}_1(t), p_1(t))_n dt = \int_0^\alpha (\hat{p}_1(t), Lp_1(t))_n dt$$
$$-(\hat{p}_1(\alpha), p_1(\alpha))_n + (\hat{p}_1(0), p_1(0))_n$$
$$= \int_0^\alpha (\hat{p}_1(t), Q_2(t)z_1(t))_n dt - (\hat{p}_1(\alpha), p_1(\alpha))_n + (\hat{p}_1(0), p_1(0))_n. \tag{3.17}$$

Subtracting from (3.16) equality (3.17), we obtain

$$0 = (\hat{p}_1(\alpha), p_1(\alpha))_n - (z_1(\alpha), \hat{\varphi}_1(\alpha))_n - (\hat{p}_1(0), p_1(0))_n + (z_1(0), \hat{\varphi}_1(0))_n + \int_0^\alpha (z_1(t), f^{(0)}(t))_n dt$$

or

$$\int_0^\alpha (z_1(t), f^{(0)}(t))_n dt + (\hat{p}_1(\alpha), p_1(\alpha))_n - (z_1(\alpha), \hat{\varphi}_1(\alpha))_n = (\hat{p}_1(0), p_1(0))_n - (z_1(0), \hat{\varphi}_1(0))_n. \tag{3.18}$$

Since

$$(z_1(0), \hat{\varphi}_1(0))_n = (\bar{B}_0 z_1(0), B_0\hat{\varphi}_1(0))_m = (\bar{B}_0 z_1(0), Q_0^{-1}\bar{B}_0\hat{p}_1(0))_m + (\bar{B}_0 z_1(0), f_0^{(0)})_m,$$
$$(\hat{p}_1(0), p_1(0))_n = (\bar{B}_0\hat{p}_1(0), B_0 p_1(0))_m = (\bar{B}_0\hat{p}_1(0), Q_0^{-1}\bar{B}_0 z_1(0))_m,$$

we can use the latter equalities, (3.18), and the fact that Q_0^{-1} is a symmetric matrix to obtain

$$(\hat{p}_1(\alpha), p_1(\alpha))_n - (z_1(\alpha), \hat{\varphi}_1(\alpha))_n = -\int_0^\alpha (z_1(t), f^{(0)}(t))_n dt - (\bar{B}_0 z_1(0), f_0^{(0)})_m. \tag{3.19}$$

Performing a similar analysis, one can prove that

$$-(p_4(\beta), \hat{p}_4(\beta))_n + (z_4(\beta), \hat{\varphi}_4(\beta))_n = -\int_\beta^T (z_4(t), f^{(0)}(t))_n dt + (\bar{B}_1 z_1(T), f_1^{(0)})_{n-m}. \tag{3.20}$$

From (3.19), (3.20), and (3.15) and the expression for $\hat{c}$, it follows

$$\widehat{\widehat{(a, \varphi(s))}} = (a, \hat{\varphi}_2(s)).$$

The theorem is proved. □

Corollary. *Function $\hat{\varphi}_2(s)$ can be taken as an estimate of solution $\varphi(s)$ of initial BVP (1.1), (1.2).*

As an example, consider the case when a vector-function $y(t) = H(t)\varphi(t) + \xi(t)$ is observed on an interval $(0, T)$, where a vector-function $\varphi(t)$ with values in $\mathbb{R}^n$ is a solution to the BVP

$$L_1\varphi = f(t), \quad \varphi(0) = f_0, \quad \varphi(T) = f_1, \tag{3.21}$$

and operator L_1 is defined by the relation

$$L_1\varphi(t) = -\varphi''(t) + q(t)\varphi(t),$$

where $q(t)$ is a positive definite $n \times n$-matrix whose entries are continuous functions on $[0, T]$.

Note that this problem has the unique classical solution if $f(t)$ is continuous on $[0, T]$ and the unique generalized solution if $f(t) \in L_2(0, T)$.

Assume that, as well as in the previous case, $H(t)$ is an $l \times n$ matrix with the entries that are continuous functions on $[\alpha, \beta]$ and $\xi(t)$ is a random vector process with zero expectation $M\xi(t)$ and unknown $l \times l$ correlation matrix $R(t, s) = M\xi(t)\xi^T(s)$. Assume also that domains V and G are given in the form (2.2) and (2.3) where matrices Q_0, Q_1, and $Q_2(t)$ entering (2.3) have dimensions $n \times n$, $f_0^{(0)} = 0$, $f_1^{(0)} = 0$, and $f^{(0)}(x) = 0$.

Write equation (3.21) as a first-order system by setting $\varphi_1(t) = \varphi'(t)$, $\varphi_2(t) = \varphi(t)$ and introducing a vector-function

$$\tilde{\varphi}(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, \quad \frac{d\tilde{\varphi}(t)}{dt} = \begin{pmatrix} \frac{d\varphi_1(t)}{dt} \\ \frac{d\varphi_2(t)}{dt} \end{pmatrix},$$

$$\tilde{f}(t) = \begin{pmatrix} f(t) \\ 0 \end{pmatrix}, \quad \tilde{f}_0 = \begin{pmatrix} 0 \\ f_0 \end{pmatrix}, \quad \tilde{f}_1 = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$$

with $2n$ components, a vector $\tilde{a} = (0, a)$ with $2n$ components, a $2n \times 2n-$matrix

$$\tilde{A}(t) = \begin{pmatrix} O_{n,n} & -q(t) \\ -E_n & O_{n,n} \end{pmatrix}$$

matrices $B_0 = B_1 = \bar{B}_0 = \bar{B}_1 = (O_{n,n}, E_n)$, and $\hat{B}_0 = \hat{B}_1 = (E_n, O_{n,n})$, and an operator

$$\tilde{L}\tilde{\varphi}(t) = -\frac{d\tilde{\varphi}(t)}{dt} + \tilde{A}^T(t)\tilde{\varphi}(t).$$

Then system (3.21) can be written as

$$\tilde{L}\tilde{\varphi}(t) = \frac{d\tilde{\varphi}(t)}{dt} + A\tilde{\varphi}(t) = \tilde{f}(t), \quad B_0\tilde{\varphi}(0) = \tilde{f}_0, \quad B_1\tilde{\varphi}(T) = \tilde{f}_1. \tag{3.22}$$

Applying Theorems 1 and 2 and performing necessary transformations in the resulting equations that are similar to (3.2) and (3.11) (in terms of the designations introduced above) we prove the following

Theorem 3.3. *The minimax estimate of expression $(a, \varphi(s))_n$ has the form*

$$(\widehat{a, \varphi(s)})_n = \int_0^s (\hat{u}_1(t), y(t))_l dt + \int_s^T (\hat{u}_2(t), y(t))_l dt = (a, \hat{\varphi}_2(s))_n,$$

where

$$\sigma^2 = (a, p_2(s))_n,$$

$\hat{u}_1(t) = Q(t)H(t)p_2(t)$, $\hat{u}_2(t) = Q(t)H(t)p_3(t)$, and vector-functions $\hat{\varphi}_2(t)$, p_2, and p_3 are determined from the solution to the equation systems

$$L_1 z_2(t) = -H^T(t)Q(t)H(t)p_2(t), \quad 0 < t < s, \quad z_2(0) = 0,$$

$$L_1 z_3(t) = -H^T(t)Q(t)H(t)p_3(t), \quad s < t < T, \quad z_2(s) - z_3(s) = a, \quad z_3(T) = 0,$$

$$L_1 p_2(t) = Q_2^{-1}(t)z_2(t), \quad 0 < t < s, \quad p_2(0) = Q_0^{-1}z_2(0),$$

$$L_1 p_3(t) = Q_2^{-1}(t)z_3(t), \quad s < t < T, \quad p_2(s) = p_3(s), \quad p_3(T) = -Q_1^{-1}z_3(T),$$

$$L_1\hat{p}_2(t) = H^T(t)Q(t)(y(t) - H(t)\hat{\varphi}_2(t)), \quad 0 < t < s, \quad \hat{p}_2(0) = 0,$$

$$L_1\hat{p}_3(t) = H^T(t)Q(t)(y(t) - H(t)\hat{\varphi}_3(t)), \quad s < t < T,$$

$$\hat{p}_3(s) = \hat{p}_2(s), \quad \hat{p}_3(T) = 0,$$

$$L_1\hat{\varphi}_2(t) = Q_2^{-1}(t)\hat{p}_2(t), \quad 0 < t < s, \quad \hat{\varphi}_2(0) = Q_0^{-1}\hat{p}_2(0),$$

$$L_1\hat{\varphi}_3(t) = Q_2^{-1}(t)\hat{p}_3(t), \quad s < t < T, \quad \hat{\varphi}_3(s) = \hat{\varphi}_2(s), \quad \hat{\varphi}_3(T) = -Q_1^{-1}\hat{p}_3(T).$$

4 Minimax estimates of solutions subject to incomplete restrictions on unknown parameters

Assume again that observations have form (2.1) and undetermined parameters f_0, f_1 and $f(t)$ belong to the domain

$$G = \{\tilde{F} = (\tilde{f}_0, \tilde{f}_1, \tilde{f}) : \int_0^T (Q_2(t)\tilde{f}(t), \tilde{f}(t))dt \le 1\}, \tag{4.1}$$

where $Q_2(t)$ is given in (2.3). The correlation function of process $\xi(t)$ belongs to domain (2.2).

Introduce the set

$$U = \{u(\cdot) : \bar{B}_0 z_1(0,u) = 0, \bar{B}_1 z_4(T,u) = 0\} \tag{4.2}$$

here $u(t) = \begin{cases} u_1(t), & \alpha < t < s, \\ u_2(t), & s < t < \beta, \end{cases}$ where $z_i(t,u), i = \overline{1,4}$, is the solution to BVP (2.6).

Lemma 4.1.

$$\sigma(u,c) = \begin{cases} \infty, & u \notin U, \\ \sigma_1(u,c), & u \in U, \end{cases}$$

where

$$\sigma_1(u,c) = \int_0^T (Q_2^{-1}\tilde{z}(t,u), \tilde{z}(t,u))dt + \int_\alpha^\beta (Q^{-1}(t)u(t), u(t))dt + c^2 = J(u) + c^2. \tag{4.3}$$

This lemma can be proved using formula (2.16).

Lemma 4.2. *U is a convex closed set in the space $L_2(\alpha, \beta)$.*

Proof. The convexity of set U is obvious. Let us prove that this set is closed.

Note that functions $z_1(0,u)$ and $z_4(T,u)$ can be represented as

$$\begin{aligned} z_1(0,u) &= a_1 + \textstyle\int_\alpha^\beta \Phi_1(t)u(t)dt, \\ z_4(T,u) &= a_2 + \textstyle\int_\alpha^\beta \Phi_2(t)u(t)dt, \end{aligned} \tag{4.4}$$

where $\Phi_1(t)$ and $\Phi_2(t)$ are known matrix functions with the elements from $L_2(\alpha,\beta)$ and a_1 and a_2 are vectors. Expression (4.4) can be obtained if we introduce a vector z_0 such that $z_1(0,u) = z_0$. Then $z_1(\alpha,u) = \Phi(\alpha,0)z_0$, where $\Phi(t,\tau)$ is a solution to the equation

$$\frac{d\Phi(t,\tau)}{dt} = -A^*(t)\Phi(t,\tau), \Phi(\tau,\tau) = E.$$

$$z_2(s,u) = \Phi(s,\alpha)z_2(\alpha,u) = \Phi(s,\alpha)z_1(\alpha,u) + \int_\alpha^s \Phi(s,\tau)H^T(\tau)u_1(\tau)d\tau =$$

$$\Phi(s,0)z_0 + \int_\alpha^s \Phi(s,\tau)H^T(\tau)u_1(\tau)d\tau.$$

Next,

$$z_4(T,u) = \Phi(T,0)z_0 + \int_\alpha^\beta \Phi(T,\tau)H^T(\tau)u(\tau)d\tau + \Phi(T,s)a.$$

Since BVP (2.6) is uniquely solvable, there exists one and only one vector z_0 satisfying the algebraic equation system

$$\begin{cases} \bar{B}_0 z_0 = 0, \\ B_1\Phi(T,0)z_0 = -\int_\alpha^\beta \Phi(t,\tau)H^T(\tau)u(\tau)d\tau - \Phi(T,s)a. \end{cases}$$

Solving this system we determine z_0 in the form

$$z_0 = b + \int_\alpha^\beta \Phi_0(\tau)u(\tau)d\tau,$$

where $\Phi_0(\tau)$ is a known matrix function continuous on $[\alpha, \beta]$ and b is a known vector. Taking into account this equality, we obtain expression (3.4). From these relationships, it follows that if a sequence $u_n(t)$ converges in $L_2(\alpha, \beta)$ to a function $u_0(t)$, then

$$\lim_{n\to\infty} \bar{B}_0 z_1(0, u_n) = \bar{B}_0 z_1(0, u_0),$$

$$\lim_{n\to\infty} \bar{B}_1 z_4(0, u_n) = \bar{B}_1 z_4(0, u_0),$$

which proves that U is a closed set. □

Assume now that U is nonempty. Then the following statement is valid.

Theorem 4.1. *There exists the unique minimax estimate of expression $(a, \varphi(s))$ which can be represented in the form (3.1) at $\hat{c} = 0$, where vector-functions $p_2(t)$ and $p_3(t)$ solve the equations*

$$L^* z_1(t) = 0, \quad 0 < t < \alpha, \quad \hat{B}_0 z_1(0) = 0,$$

$$\begin{gathered}
L^* z_2(t) = -H^T(t)Q(t)H(t)p_2(t), \quad \alpha < t < s, \quad z_2(\alpha) = z_1(\alpha), \\
L^* z_3(t) = -H^T(t)Q(t)H(t)p_3(t), \quad s < t < \beta, \quad z_3(s) = z_2(s) - a, \\
L^* z_4(t) = 0, \quad \beta < t < T, \quad z_4(\beta) = z_3(\beta), \\
\hat{B}_1 z_4(T) = 0, \quad \bar{B}_0 z_1(0), \quad \bar{B}_1 z_4(T) = 0, \\
L p_1(t) = Q_2^{-1}(t) z_1(t), \quad 0 < t < \alpha, \\
L p_2(t) = Q_2^{-1}(t) z_2(t), \quad \alpha < t < s, \quad p_2(\alpha) = p_1(\alpha), \\
L p_3(t) = Q_2^{-1}(t) z_3(t), \quad s < t < \beta, \quad p_3(s) = p_2(s), \\
L p_4(t) = Q_2^{-1}(t) z_4(t), \quad s < t < \beta.
\end{gathered} \tag{4.5}$$

Proof. Similarly to Theorem 3.1 one can show that for $u \in U$ the following equality holds

$$\sigma(u, c) = J(u) + c^2,$$

where

$$\begin{aligned}
J(u) = &\int_0^\alpha (Q_2^{-1}(t) z_1(t,u), z_1(t,u))_n dt + \int_\alpha^s (Q_2^{-1}(t) z_2(t,u), z_2(t,u))_n dt \\
&+ \int_s^\beta (Q_2^{-1}(t) z_3(t,u), z_3(t,u))_n dt + \int_\beta^T (Q_2^{-1}(t) z_4(t,u), z_4(t,u))_n dt \\
&+ \int_\alpha^s (Q^{-1}(t) u_1(t), u_1(t)) dt + \int_s^\beta (Q^{-1}(t) u_2(t), u_2(t)) dt,
\end{aligned}$$

and $z_i(t, u), i = \overline{1, 4}$, are solutions to equations (4.5) at $\bar{B}_0 z_1(0, u) = 0$ and $\bar{B}_1 z_4(T, u) = 0$. $J(u)$ is a strictly convex lower semicontinuous functional on a closed convex set U and $\lim_{\|u\|\to\infty} J(u) = \infty$. Therefore there exists one and only one vector $\hat{u}$ such that $\min_{u\in U} J(u) = J(\hat{u})$. This vector can be determined from the relationship

$$\frac{d}{d\tau} J_\mu(\hat{u}_1 + \tau v_1, \hat{u}_2 + \tau v_2)\bigg|_{\tau=0} \equiv 0, \quad \forall v = (v_1, v_2) \in H,$$

where

$$J_\mu(u) = J(u_1, u_2) + (\mu_1, \bar{B}_0 z_1(0, u)) + (\mu_2, \bar{B}_1 z_4(T, u)),$$

$\mu = (\mu_1, \mu_2)$, $\mu_1 \in \mathbb{R}^m$, and $\mu_2 \in \mathbb{R}^{n-m}$ are Lagrange multipliers.

Further analysis is similar to the proof of Theorem 3.1. □

Let vector-functions $\hat{p}_1(t), \hat{p}_2(t), \hat{p}_3(t), \hat{p}_4(t), \hat{\varphi}_1(t), \hat{\varphi}_2(t), \hat{\varphi}_3(t), \hat{\varphi}_4(t)$ be solutions to the system

$$
\begin{gathered}
L^*\hat{p}_1(t) = 0, \quad 0 < t < \alpha, \\
L^*\hat{p}_2(t) = H^T(t)Q(t)[y(t) - H(t)\hat{\varphi}_2(t)), \quad \alpha < t < s, \\
L^*\hat{p}_3(t) = H^T(t)Q(t)[y(t) - H(t)\hat{\varphi}_3(t)), \quad s < t < \beta, \\
L^*\hat{p}_4(t) = 0, \quad \beta < t < T, \\
L\hat{\varphi}_1(t) = Q_2^{-1}(t)\hat{p}_1(t), \quad 0 < t < \alpha, \\
L\hat{\varphi}_2(t) = Q_2^{-1}(t)\hat{p}_2(t), \quad \alpha < t < s, \\
L\hat{\varphi}_3(t) = Q_2^{-1}(t)\hat{p}_3(t), \quad s < t < \beta, \\
L\hat{\varphi}_4(t) = Q_2^{-1}(t)\hat{p}_4(t), \quad \beta < t < T, \\
\hat{B}_0\hat{p}_1(0) = 0, \quad \hat{p}_2(\alpha) = \hat{p}_1(\alpha), \quad \hat{p}_3(s) = \hat{p}_2(s), \quad \hat{p}_4(\beta) = \hat{p}_3(\beta), \\
\hat{B}_1\hat{p}_1(T) = 0, \quad \bar{B}_0\hat{p}_1(0) = 0, \quad \bar{B}_1\hat{p}_4(T) = 0, \\
\hat{\varphi}_2(\alpha) = \hat{\varphi}_1(\alpha), \quad \hat{\varphi}_3(s) = \hat{\varphi}_2(s) \quad \hat{\varphi}_4(\beta) = \hat{\varphi}_3(\beta).
\end{gathered} \tag{4.6}
$$

Theorem 4.2. *Assume that for any vector $a \in \mathbb{R}^n$ set U is nonempty. Then system (4.6) is uniquely solvable and the equality*

$$\widehat{(a, \varphi(s))} = (a, \hat{\varphi}_2(s))$$

holds

Proof. Introduce functions $\hat{p}_i(t, v)$ as solutions to the BVP

$$
\begin{gathered}
L^*\hat{p}_1(t, v) = 0, \quad 0 < t < \alpha, \\
L^*\hat{p}_2(t, v) = d(t) - H^T(t)v_1(t), \quad \alpha < t < s, \\
L^*\hat{p}_3(t, v) = d(t) - H^T(t)v_2(t), \quad s < t < \beta, \\
L^*\hat{p}_4(t, v) = 0, \quad \beta < t < T, \\
\hat{B}_0\hat{p}_1(0, v) = 0, \quad \hat{B}_1\bar{p}_4(T, v) = 0, \\
\hat{p}_2(\alpha, v) = \hat{p}_1(\alpha, v), \quad \hat{p}_3(s, v) = \hat{p}_2(s, v), \quad \hat{p}_4(\beta, v) = \hat{p}_3(\beta, v),
\end{gathered}
$$

where $d(t) = H^T(t)Q(t)y(t)$.

Define a set

$$U_1 = \{v : \bar{B}_0\hat{p}_1(0, v) = 0, \bar{B}_1\hat{p}_4(T, v) = 0\}.$$

Since U is nonempty, the same is valid for U_1 for any vector a. Similarly to the case of U, one can show that U_1 is a convex closed set. Denote by $J_1(v)$ the functional of the form

$$
\begin{aligned}
J_1(v) = &\int_0^\alpha (Q_2^{-1}(t)\hat{p}_1(t, v), \hat{p}_1(t, v))dt + \int_\alpha^s (Q_2^{-1}(t)\hat{p}_2(t, v), \hat{p}_2(t, v))dt \\
&+ \int_s^\beta (Q_2^{-1}(t)\hat{p}_3(t, v), \hat{p}_3(t, v))dt + \int_\beta^T (Q_2^{-1}(t)\hat{p}_4(t, v), \hat{p}_4(t, v))dt \\
&+ \int_\alpha^s (Q^{-1}(t)v_1(t), v_1(t))dt + \int_s^\beta (Q^{-1}(t)v_2(t), v_2(t))dt.
\end{aligned}
$$

One can show, following Theorem 4.1, that on set U_1 there is one and only one point of minimum of functional $J_1(v)$, namely,

$$\hat{v}_1(t) = Q^{-1}(t)H(t)\hat{\varphi}_2(t), \quad \hat{v}_2(t) = Q^{-1}(t)H(t)\hat{\varphi}_3(t),$$

where functions $\hat{\varphi}_2(t)$ and $\hat{\varphi}_3(t)$ are determined from system (4.6). The proof of the equality

$$\widehat{(a, \varphi(s))}_n = (a, \hat{\varphi}_2(s))_n$$

is similar to that in Theorem 3.2. $\square$

5 Elimination technique in minimax estimation problems

Assume that a vector-function

$$y_1(t) = C_{11}\varphi(t) + C_{12}\varphi'(t) + \xi_1(t), \quad y_2(t) = C_{21}\varphi(t) + C_{22}\varphi'(t) + \xi_2(t) \tag{5.1}$$

is observed on interval $(0,1)$. Here $C = C_{ij}(t)$ is an $m \times n$ matrix with the entries continuous on $[0,1]$; $\xi_i(t)$ are vector random processes continuous in the mean square sense and such that $M\xi_i(t) = 0$, $i = 1,2$; function $\varphi(t)$ is a solution to the BVP

$$\varphi''(t) = A(t)\varphi(t) + B(t)f(t), \quad \varphi'(0) = 0, \quad \varphi(1) = 0, \tag{5.2}$$

where $A(t)$ and $B(t)$ are, respectively, $n \times n$ and $n \times r$ matrices with the entries continuous on $[0,1]$ and $A(t)$ is a symmetric nonnegative definite matrix; and $f(t)$ is a square integrable vector-function on $(0,1)$.

Set $\varphi = \varphi_1$ and $\varphi_1' = \varphi_2$ and rewrite (5.2) as an equation system

$$\varphi_1' = \varphi_2, \quad \varphi_2' = A(t)\varphi_1 + B(t)f, \quad \varphi_2(0) = \varphi_1(1) = 0. \tag{5.3}$$

Reduce the solution of BVP (5.3) to the solution of a Cauchy problem using elimination.

We look for function $\varphi_2(t)$ in the form $\varphi_2(t) = P(t)\varphi_1(t) + \psi(t)$, where matrix $P(t)$ and vector-function $\psi(t)$ is chosen so that

$$\varphi_2'(t) = A(t)\varphi_1(t) + B(t)f(t).$$

We have

$$\varphi_2'(t) = P'(t)\varphi_1(t) + P(t)\varphi_1'(t) + \psi'(t) = P'(t)\varphi_1(t) + P(t)\varphi_2(t) + \psi'(t) =$$
$$= P'(t)\varphi_1(t) + P(t)[P(t)\varphi_1(t) + \psi(t)] + \psi'(t) = A(t)\varphi_1(t) + B(t)f(t);$$

therefore, functions $P(t)$ and $\psi(t)$ must satisfy the equations

$$P'(t) + P^2(t) = A(t),\ P(0) = 0,\ \psi'(t) + P(t)\psi(t) = B(t)f(t),\ \psi(0) = 0. \tag{5.4}$$

Thus we have reduced BVP (5.2) to the Cauchy problem for functions $P(t)$, $\psi(t)$, and $\varphi_1(t)$.

Note that for function $P(t)$ a Riccati equation is obtained which is uniquely solvable on any finite interval; in addition, there is one and only one function $\psi(t)$ which is absolutely continuous and satisfies the corresponding equation almost everywhere.

Using the notations introduced above, we can write the expressions for functions $y_1(t)$ and $y_2(t)$ as

$$y_1(t) = [C_{11}(t) + C_{12}(t)P(t)]\varphi_1(t) + C_{12}(t)\psi(t) + \xi_1(t),$$
$$y_2(t) = [C_{21}(t) + C_{22}(t)P(t)]\varphi_1(t) + C_{22}(t)\psi(t) + \xi_2(t),$$

or

$$y(t) = H(t)x(t) + \xi(t), \tag{5.5}$$

where

$$H(t) = (H_{ij}(t))_{i,j=1,2}, \quad H_{11} = C_{11} + C_{12}P,$$
$$H_{12} = C_{12}, \quad H_{21} = C_{21} + C_{22}P, \quad H_{22} = C_{22},$$
$$x(t) = \begin{pmatrix} \varphi_1(t) \\ \psi(t) \end{pmatrix}, \quad \xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \end{pmatrix},$$

and function $x(t)$ is a solution to the equation

$$\frac{dx}{dt} = A_1(t)x(t) + B_1(t)f(t), \quad \varphi_1(1) = 0, \quad \psi(0) = 0, \tag{5.6}$$

where

$$A_1(t) = \begin{pmatrix} P(t) & E \\ 0 & -P(t) \end{pmatrix}, \quad B_1(t) = \begin{pmatrix} 0 \\ B(t) \end{pmatrix}.$$

Denote by V a class of random processes $\tilde{\xi}(t)$ whose correlation matrices $\tilde{R}(t,s) = M\tilde{\xi}(t)\tilde{\xi}^T(s)$ satisfy the inequality

$$\int_0^1 q_1^2(t) Sp\, \tilde{R}(t,t)dt \le 1,$$

here $q_1(t)$ is a continuous function on $[0,T]$ such that $|q_1(t)| \ge \mathrm{q} > 0$ where q is a certain number. Assume in what follows that correlation matrix $R(t,s)$ of process $\xi(t)$ belongs to V.

Assume also that function $f(t)$ belongs to a set

$$G = \left\{ \tilde{f} : \int_0^1 (Q\tilde{f}, \tilde{f})dt \le 1 \right\}, \tag{5.7}$$

where Q is a positive definite matrix.

Our problem is to find a minimax estimate of expression $(a_1, \varphi_1(s)) + (a_2, \varphi_2(s))$ using observations (5.1). Since $(a_1, \varphi_1(s)) + (a_2, \varphi_2(s)) = (a_1 + Pa_2, \varphi_1(s)) + (a_2, \psi(s)) = (b, x(s))$ where $b = \begin{pmatrix} a_1 + Pa_2 \\ a_2 \end{pmatrix}$, there is a one-to-one correspondence that couples the estimates.

We look for an estimate[2] of expression $(b, x(s))$ in the class of estimates of the form

$$\widehat{(b, x(s))} = \int_0^s (u_1(t), y(t))dt + \int_s^1 (u_2(t), y(t))dt + d \tag{5.8}$$

linear with respect to observations where $u_1(t)$ and $u_2(t)$ are square integrable functions on $(0,s)$ and $(s,1)$, respectively, and $d \in \mathbb{R}^1$.

As before, we look for minimax estimate $\widehat{\widehat{(b, x(s))}} = \int_0^s (\hat{u}_1(t), \tilde{y}(t))dt + \int_s^1 (\hat{u}_2(t), \tilde{y}(t))dt + \hat{d}$ of inner product $(b, x(s))$ in the class of linear estimates of the form $\widehat{(b, x(s))} = \int_0^s (u_1(t), y(t))dt + \int_s^1 (u_2(t), y(t))dt + d$. For this estimate vector-function $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))$ and constant $\hat{d}$ are determined from the condition

$$\inf_{u_1 \in L^2(0,s), u_2 \in L^2(s,1), d \in \mathbb{R}} \sup_{\tilde{\xi} \in V, \tilde{f} \in G} M[(b, \tilde{x}(s)) - \widehat{(b, \tilde{x}(s))}]^2 = \sup_{\tilde{\xi} \in V, \tilde{f} \in G} M[(b, \tilde{x}(s)) - \widehat{\widehat{(b, \tilde{x}(s))}}]^2, \tag{5.9}$$

where $\tilde{x}$ is a solution to problem (5.2) at $f = \tilde{f}$, $\widehat{(b, \tilde{x}(s))} = \int_0^s (u_1(t), \tilde{y}(t))dt + \int_s^1 (u_2(t), \tilde{y}(t))dt + d$, $\tilde{y}(t) = H(t)\tilde{x}(t) + \tilde{\xi}(t)$.

Introduce functions $z_1(s)$ and $z_2(s)$ as solutions to the equations

$$\begin{gathered} z_1' = -A_1^T z_1 + H^T u_1, \quad z_{11}(0) = 0, \quad 0 < t < s, \\ z_2' = -A_1^T z_2 + H^T u_2, \quad z_{22}(1) = 0, \quad s < t < 1, \\ -z_2(s) + z_1(s) = b. \end{gathered} \tag{5.10}$$

[2]In this section $(\cdot, \cdot)$ will denote the inner product in the corresponding Euclidean space

Proposition 5.1. *Let $z_1(s)$ and $z_2(s)$ be solutions to the BVP for equations (5.10); then the following equality holds*

$$\sup_{\tilde{\xi}\in V,\tilde{f}\in G} M[(b,\tilde{x}(s)) - \widehat{(b,\tilde{x}(s))}]^2 = \int\limits_0^s (Q_1 z_1, z_1)dt + \int\limits_s^1 (Q_1 z_2, z_2)dt +$$

$$+\int\limits_0^s q_1^{-2}(u_1,u_1)dt + \int\limits_s^1 q_1(u_2,u_2)dt + d^2, \tag{5.11}$$

where

$$Q_1 = B_1 Q^{-1} B_1^T.$$

The proof of this statement is similar to the proof of the corresponding assertion from Section 2.1.

Remark. The BVP is uniquely solvable in the class of absolutely continuous functions.

Indeed, writing the equations for the components of vectors z_1 and z_2 we obtain an equation system

$$\begin{aligned} z_{11}' &= -Pz_{11} + (H^T u_1)_1, \quad z_{11}(0) = 0, \\ z_{12}' &= -z_{11} + Pz_{12} + (H^T u_1)_1, \quad -z_{21}(s) + z_{11}(s) = b_1, \\ z_{21}' &= -Pz_{21} + (H^T u_2)_1, \quad -z_{22}(s) + z_{12}(s) = b_2, \\ z_{22}' &= -z_{21} + Pz_{22} + (H^T u_2)_2, \quad z_{22}(1) = 0. \end{aligned} \tag{5.12}$$

For function z_{11} we have a Cauchy problem; solving this problem we find $z_{21}(s)$ and, consequently, $z_{22}(s)$. Solving the Cauchy problem for $z_{21}(t)$, we obtain this function for any $t \in (s,1)$. Function $z_{12}(t)$ can be determined in a similar manner.

Proposition 5.2. *Let functions $u_1(t)$ and $u_2(t)$ have the form $u_1(t) = p_1(t)H(t)z_1(t)$ and $u_2(t) = p_2(t)H(t)z_2(t)$, where $p_1(t)$ and $p_2(t)$ are continuous, respectively, on $[0,s]$ and $[s,1]$. Then the following representation is valid*

$$\int\limits_0^s (u_1(t), y(t))dt + \int\limits_s^1 (u_2(t), y(t))dt = (b, \hat{x}_1(s)), \tag{5.13}$$

where $\hat{x}_1(s)$ is determined from the solution to the equation system

$$\begin{aligned} \hat{x}_1'(t) &= A_1\hat{x}_1 + p_1(t)H^T(y(t) - H\hat{x}_1(t)), \\ \hat{x}_2'(t) &= A_1\hat{x}_2 + p_2(t)H^T(y(t) - H\hat{x}_2(t)), \\ \hat{x}_1(s) &= \hat{x}_2(s), \quad \hat{x}_{12}(0) = \hat{x}_{21}(1) = 0. \end{aligned} \tag{5.14}$$

Proof. Since $u_2(t) = p_2(t)H(t)z_2(t)$, we have

$$\int\limits_s^1 (u_2(t), y(t))dt = \int\limits_s^1 (z_2(t), p_2(t)H^T(t)y(t))dt.$$

Let $\hat{x}_2(s)$ be a solution determined from system (5.14). Multiplying both sides of equations (5.14) by $z_2(t)$ and integrating from s to 1, we obtain

$$\int\limits_s^1 (z_2, p_2 H^T y)dt = \int\limits_s^1 (\hat{x}_2', z_2)dt - \int\limits_s^1 (A_1\hat{x}_2, z_2)dt + \int\limits_s^1 (p_2 H^T H\hat{x}_2, z_2)dt,$$

however,

$$\int_s^1 (\hat{x}_2', z_2)dt = -\int_s^1 (\hat{x}_2, z_2')dt + (\hat{x}_2(1), z_2(1)) - (\hat{x}_2(s), z_2(s)) =$$

$$= \int_s^1 (\hat{x}_2, A_1^T z_2)dt - \int_s^1 (\hat{x}_2, p_2 H^T H z_2)dt.$$

Thus

$$\int_s^1 (z_2, p_2 H^T y)dt = -(\hat{x}_2(s), z_2(s)).$$

In a similar way, we can show that

$$\int_0^s (u_1, y)dt = \int_0^s (z_1, p_1 H^T y)dt = (\hat{x}_1(s), z_1(s)).$$

The desired representation follows now from the latter relationship and the equalities $\hat{x}_1(s) = \hat{x}_2(s)$, $z_1(s) - z_2(s) = b$. □

Introduce functions $p_1(t)$, $p_2(t)$, $\hat{x}_1(t)$, and $\hat{x}_1(t)$ as solutions to the initial value problems

$$\begin{gathered} z_1' = -A_1^T z_1 + H^T q_1^2 H p_1, \quad z_{11}(0) = 0, \\ z_2' = -A_1^T z_2 + H^T q_1^2 H p_2, \quad z_{22}(1) = 0, \\ z_1(s) - z_2(s) = b, \\ p_1' = A_1 p_1 + Q_1 z_1, \quad p_{12}(0) = 0, \\ p_2' = A_1 p_2 + Q_1 z_2, \quad p_{21}(1) = 0, \\ p_1(s) = p_2(s), \end{gathered} \tag{5.15}$$

$$\begin{gathered} \hat{x}_1' = A_1 \hat{x}_1 + Q_1 \hat{p}_1, \quad \hat{x}_{12}(0) = 0, \\ \hat{x}_2' = A_1 \hat{x}_2 + Q_1 \hat{p}_2, \quad \hat{x}_{21}(1) = 0, \\ \hat{x}_1(s) = \hat{x}_2(s), \\ -\hat{p}_1' = A_1^T \hat{p}_1 + H^T q_1^2 (y - H\hat{x}_1), \quad \hat{p}_{11}(0) = 0, \\ -\hat{p}_2' = A_1^T \hat{p}_2 + H^T q_1^2 (y - H\hat{x}_2), \quad \hat{p}_{22}(1) = 0, \\ \hat{p}_1(s) = \hat{p}_2(s). \end{gathered} \tag{5.16}$$

The following statement is valid.

Proposition 5.3. *Let the set G have form (5.7). The the minimax estimate admits the representation*

$$\widehat{\widehat{(b, x(s))}} = \int_0^s (q_1^2 H p_1, y)dt + \int_s^1 (q_1^2 H p_2, y)dt = (b, \hat{x}_1(s)), \tag{5.17}$$

where functions p_1, p_2, and $\hat{x}_1$ are determined from the solution to equation systems (5.15), (5.16).

The proof of this statement is similar to the proof of the corresponding assertion from Section 2.1. Write equations (5.16) in a detailed form, row by row,

$$\hat{x}'_{11} = P\hat{x}_{11} + \hat{x}_{11}, \quad \hat{x}_{12}(0) = 0,$$

$$\hat{x}'_{12} = -P\hat{x}_{12} + \tilde{Q}\hat{p}_{12}, \quad \hat{x}_{21}(1) = 0,$$

$$\hat{x}'_{21} = P\hat{x}_{21} + \hat{x}_{22}, \quad \hat{x}_{11}(s) = \hat{x}_{21}(s),$$

$$\hat{x}'_{22} = -P\hat{x}_{22} + \tilde{Q}\hat{p}_{22}, \quad \hat{x}_{12}(s) = \hat{x}_{22}(s),$$

where $\tilde{Q} = BQ_1B^T$. Let us go back to equation (5.6). Since the equation for function $\psi(t)$ has the form

$$\psi'(t) = -P(t)\psi(t) + B(t)f(t), \quad \psi(0) = 0,$$

the Cauchy formula yields

$$\psi(1) = \int\limits_0^1 \Phi(1,t)B(t)f(t)dt = \mathcal{D}f,$$

where $\Phi(s,t)$ is a solution to the equation

$$\frac{\partial\Phi(s,t)}{\partial s} = P(s)\Phi(s,t),$$

$$\Phi(t,t) = E.$$

Below, we will study estimates of solutions to equation (5.6) subject to the conditions $\varphi_1(1) = 0$, $\psi(1) = \mathcal{D}f$. Let the function $f(t)$ belong to a bounded subset G of $L_2(0,1)$.

Introduce functions $z_1(t;u)$ and $z_2(t;u)$ as solutions to the equations

$$z'_1(\cdot;u) = -A_1^T z_1(\cdot;u) + H^T u_1, \quad z_1(0) = 0,$$

$$z'_2(\cdot;u) = -A_1^T z_2(\cdot;u) + H^T u_2, \quad z_1(s;u) - z_2(s;u) = b. \tag{5.18}$$

Proposition 5.4. *Let $z_1(t;u)$ and $z_2(t;u)$ be solutions to the equation system (5.18). Then*

$$\sup_{\tilde{\xi}\in V, \tilde{f}\in G} M((b,\tilde{x}(s)) - \widehat{(b,\tilde{x}(s))})^2 = \sup_{\tilde{f}\in G}\Bigg(\int_0^s \left(z_1(t;u) + \Phi^T(1,t)z_1(1;u), B(t)\tilde{f}(t)\right) dt +$$

$$+\int_s^1 \left(z_2(t;u) + \Phi^T(1,t)z_2(1;u), B(t)\tilde{f}(t)\right) dt - d\Bigg)^2 + \int_0^s q_1^{-2}(t)(u_1(t),u_1(t))dt +$$

$$+\int_s^1 q_1^{-2}(t)(u_2(t),u_2(t))dt.$$

This statement can be easily proved using the methods set forth in Section 2.1.

We will call minimax estimate $\widehat{\widehat{(b,x(s))}}$ $\mathcal{U}$ optimal if it has the form

$$\widehat{\widehat{(b,x(s))}} = \int_0^s (\hat{u}_1(t),y(t))\,dt + \int_s^1 (\hat{u}_2(t),y(t))\,dt + \hat{d},$$

where functions $\hat{u}_1$ and $\hat{u}_2$ and number $\hat{d}$ are determined from the condition

$$\inf_{(u_1,u_2)\in\mathcal{U}, d\in\mathbb{R}} \sup_{\tilde{\xi}\in V, \tilde{f}\in G} M\left((b,\tilde{x}(s)) - \widehat{(b,\tilde{x}(s))}\right)^2 = \sup_{\tilde{\xi}\in V, \tilde{f}\in G} M\left((b,\tilde{x}(s)) - \widehat{\widehat{(b,\tilde{x}(s))}}\right)^2$$

and $\mathcal{U}$ is a subset of $L_2(0,1)$.

Let the set $\mathcal{U}$ be given by $\mathcal{U} = \{(u_1,u_2) : z_2(1;u) = 0\}$, and G is determined by formula (5.7).

Proposition 5.5. *Assume that set $\mathcal{U}$ is not empty. Then $\hat{d} = 0$, $\hat{u}_1(t) = q_1(t)H(t)p_1(t)$, and $\hat{u}_2(t) = q_1(t)H(t)p_2(t)$, where $p_1(t)$ and $p_2(t)$ are determined from the solution to the equation system*

$$z_1' = -A_1^T z_1 + H^T \hat{u}_1, \quad z_1(0) = 0,$$

$$z_2' = -A_1^T z_2 + H^T \hat{u}_2, \quad z_1(s) - z_2(s) = b, \quad z_2(1) = 0,$$

$$p_1' = A_1 p_1 + Q_1 z_1, \quad 0 < t < s,$$

$$p_2' = A_1 p_2 + Q_1 z_2, \quad s < t < 1, \quad p_1(s) = p_2(s). \tag{5.19}$$

Proof. If set U is not empty, then from Proposition 4, it follows that for $(u_1, u_2) \in U$,

$$\sup_{\tilde{\xi} \in V, \tilde{f} \in G} M((b, \tilde{x}(s)) - (\widehat{b, \tilde{x}(s)}))^2 =$$

$$\sup_{\tilde{f} \in G} \left[\int_0^s (z_1(t;u), B(t)\tilde{f}(t))dt + \int_s^1 (z_2(t;u), B(t)\tilde{f}(t))dt - d \right]^2 +$$

$$+ \int_0^s q_1^{-2}(t)(u_1(t), u_1(t))dt + \int_s^1 q_1^{-2}(t)(u_2(t), u_2(t))dt = J(u,d).$$

Taking into account the definition of set G, we obtain

$$J(u,d) = \int_0^s (Q_1 z_1(\cdot;u), z_1(\cdot;u))dt + \int_s^1 (Q_1 z_2(\cdot;u), z_2(\cdot;u))dt +$$

$$+ \int_0^s q_1^{-2}(t)(u_1(t), u_1(t))dt + \int_s^1 q_1^{-2}(t)(u_2(t), u_2(t))dt + d^2.$$

This expression yields

$$\inf_{(u_1,u_2) \in U, d \in \mathbb{R}} J(u,d) = \inf_{(u_1,u_2) \in U} J(u,0).$$

It is easy to see that there exists a unique function $\hat{u}(t) = (\hat{u}_1(t), \hat{u}_2(t))$ such that

$$J(\hat{u}, 0) = \inf_{u=(u_1,u_2) \in U} J(u,0).$$

Set $z_1(t) := z_1(t;\hat{u})$ and $z_2(t) := z_2(t;\hat{u})$. Finding conditional extremum of functional $J(u,0)$ we obtain that for the function $\hat{u}(t)$ at which the infimum is attained, the following equalities hold

$$\hat{u}_1(t) = q_1(t)H(t)p_1(t), \quad \hat{u}_2(t) = q_1(t)H(t)p_2(t),$$

where $p_1(t)$ and $p_2(t)$ are determined from the solution to the equation system (5.19). The theorem is proved. $\square$

Introduce functions $\hat{p}_1(t)$, $\hat{p}_2(t)$, $\hat{x}_1(t)$, and $\hat{x}_2(t)$ as solutions to the equation system

$$-\hat{p}_1' = A_1^T \hat{p}_1 + q_1 H^T (y(t) - H\hat{x}_1(t)), \quad \hat{p}_1(0) = 0,$$

$$-\hat{p}_2' = A_1^T \hat{p}_2 + q_1 H^T (y(t) - H\hat{x}_2(t)), \quad \hat{p}_2(1) = 0, \quad \hat{p}_1(s) = \hat{p}_2(s),$$

$$\hat{x}_1' = A_1 \hat{x}_1 + Q_1 \hat{p}_1, \quad 0 < t < s,$$

$$\hat{x}_2' = A_1 \hat{x}_2 + Q_1 \hat{p}_2, \quad s < t < 1, \quad \hat{x}_1(s) = \hat{x}_2(s).$$

Proposition 5.6. *The minimax estimate admits the representation $(\widehat{\widehat{b, x(s)}}) = (b, \hat{x}_1(s))$.*

Proof. Since $\hat{u}_1(s) = q_1(t)H(t)p_1(t)$, we have

$$\int_0^s (\hat{u}_1, y)dt = \int_0^s (q_1 H p_1, y)dt = \int_0^s (p_1, q_1 H^T y)dt =$$

$$= -\int_0^s (p_1, \hat{p}_1')dt - \int_0^s (p_1, A_1^T \hat{p}_1)dt + \int_0^s (p_1, q_1 H^T H \hat{x}_1)dt =$$

$$= \int_0^s (p_1', \hat{p}_1)dt - \int_0^s (A_1^T p_1, \hat{p}_1)dt - (p_1(s), \hat{p}_1(s)) + \int_0^s (p_1, q_1 H^T H \hat{x}_1)dt =$$

$$= \int_0^s (z_1, Q_1 \hat{p}_1)dt - (p_1(s), \hat{p}_1(s)) + \int_0^s (p_1, q_1 H^T H \hat{x}_1)dt,$$

$$\int_0^s (z_1, Q_1 \hat{p}_1)dt = \int_0^s (\hat{x}_1' - A_1 \hat{x}_1, z_1)dt =$$

$$= -\int_0^s (\hat{x}_1, z_1' + A_1^T z_1)dt + (\hat{x}_1(s), z_1(s)) - \int_0^s (p_1, q_1 H^T H \hat{x}_1)dt.$$

The latter relationships yield $\int_0^s (\hat{u}_1, y)dt = (\hat{x}_1(s), z_1(s))$.

In a similar manner, we can show that

$$\int_s^1 (\hat{u}_2, y)dt = -(\hat{x}_2(s), z_2(s)) = -(\hat{x}_1(s), z_2(s)).$$

Therefore,

$$\int_0^s (\hat{u}_1, y)dt + \int_s^1 (\hat{u}_2, y)dt = (b, \hat{x}_1(s)),$$

The proposition is proved. □

6 Minimax estimation of the solutions to the boundary value problems from point observations

In this chapter we study minimax estimation problems in the case of point observations and propose constructive methods for obtaining minimax estimates.

Let t_i, $i = \overline{1, N}$ be a given system of points on an interval $(0, T)$. Set $t_0 = 0$ and $t_{N+1} = T$. The problem is to estimate the expression

$$(a, \varphi(s))_n = \sum_{i=1}^n a_i \varphi_i(s), \tag{6.1}$$

from observations of the form

$$y_i = \varphi(t_i) + \xi_i, \ i = \overline{1, N}, \tag{6.2}$$

over the state φ of a system described by BVP (1.1), (1.2) in the class of estimates

$$\widehat{(a,\varphi(s))}_n = \sum_{i=1}^{N}(u_i, y_i)_n + c \tag{6.3}$$

linear with respect to observations (6.2); here $s \in (t_{i_0-1}, t_{i_0})$, $i_0 \in \{1,\ldots,N+1\}$. The assumptions are as follows[3] $F = (f_0, f_1, f(\cdot)) \in G$, $\xi := (\xi_1,\ldots,\xi_N) \in V$, where ξ_i are errors in estimations (6.2) that are realizations of random vectors $\xi_i = \xi_i(\omega) \in \mathbb{R}^n$ and V denotes the set of random elements $\tilde\xi = (\tilde\xi_1,\ldots,\xi_N)$ whose components ξ_i has integrable second moments $M\tilde\xi_i^2$, zero means $M\tilde\xi_i = 0$,, and correlation matrices $\tilde R_i = M\tilde\xi_i\tilde\xi_i^T$ satisfying the condition

$$\sum_{i=1}^{N} Sp\,[\tilde Q_i \tilde R_i] \le 1, \tag{6.4}$$

where $\tilde Q_i$ are positive definite $n\times n$ matrices, $\operatorname{Sp} B$ denotes the trace of the matrix $B = \{b_{ij}\}_{i,j=1}^{l}$, i.e., the quantity $\sum_{i=1}^{l} b_{ii}$, $u_i \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

Set $u := (u_1,\ldots,u_N) \in \mathbb{R}^{N\times n}$.

Definition. *The estimate*

$$\widehat{\widehat{(a,\varphi(s))}}_n = \sum_{i=1}^{N}(\hat u_i, y_i)_n + \hat c,$$

in which vectors $\hat u_i$, and a number $\hat c$ are determined from the condition

$$\sup_{\tilde F\in V, \tilde\xi\in G} M|(a,\tilde\varphi(s))_n - \widehat{(a,\tilde\varphi(s))}_n|^2 \to \inf_{u\in\mathbb{R}^{N\times n},\, c\in\mathbb{R}}, \tag{6.5}$$

where

$$\widehat{(a,\tilde\varphi(s))}_n = \sum_{i=1}^{N}(u_i, \tilde y_i)_n + c, \tag{6.6}$$

$$\tilde y_i = \tilde\varphi(t_i) + \tilde\eta_i, \quad i = \overline{1,N}, \tag{6.7}$$

and $\tilde\varphi(t)$ is the solution to the BVP (1.1), (1.2) at $f = \tilde f$, $f_0 = \tilde f_0$, and $f_1 = \tilde f_1$, will be called the minimax estimate of expression (6.1).

The quantity

$$\sigma := \{\sup_{\tilde F\in G, \tilde\xi\in V} M|(a,\varphi(s))_n - \widehat{\widehat{(a,\varphi(s))}}_n|^2\}^{1/2} \tag{6.8}$$

will be called the error of the minimax estimation of $(a,\varphi(s))_n$.

Let again $t_0 = 0$, $t_{N+1} = T$, and $s \in (t_{i_0-1}, t_{i_0})$, $i_0 = \overline{1,N+1}$. For any fixed $u := (u_1,\ldots,u_N) \in \mathbb{R}^{N\times n}$ introduce vector-functions $z_1(\cdot;u) \in H^1(t_0,t_1)^n$, $\ldots$, $z_{i_0-1}(\cdot;u) \in H^1(t_{i_0-1},t_{i_0})^n$, $z_{i_0}^{(1)}(\cdot;u) \in H^1(t_{i_0},s)^n$, $z_{i_0}^{(2)}(\cdot;u) \in H^1(s,t_{i_0})^n$, $z_{i_0+1}(\cdot;u) \in H^1(t_{i_0},t_{i_0+1})^n$, $\ldots$, $z_{N+1}(\cdot;u) \in H^1(t_{i_N},t_{i_N+1})^n$, as solution to the

[3] Set G is defined by (2.3)

following BVP

$$
\begin{gathered}
L^*z_1(t;u)=0,\quad 0<t<t_1,\quad \hat{B}_0z_1(0;u)=0,\\
L^*z_2(t;u)=0,\quad t_1<t<t_2,\quad z_2(t_1;u)=z_1(t_1;u)+u_1,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
L^*z_{i_0-1}(t;u)=0,\quad t_{i_0-2}<t<t_{i_0-1},\quad z_{i_0-1}(t_{i_0-2};u)=z_{i_0-2}(t_{i_0-2};u)+u_{i_0-2},\\
L^*z^{(1)}_{i_0}(t;u)=0,\quad t_{i_0-1}<t<s,\quad z^{(1)}_{i_0}(t_{i_0-1};u)=z_{i_0-1}(t_{i_0-1};u)+u_{i_0-1},\\
L^*z^{(2)}_{i_0}(t;u)=0,\quad s<t<t_{i_0},\quad z^{(2)}_{i_0}(s;u)=z^{(1)}_{i_0}(s;u)-a,\\
L^*z_{i_0+1}(t;u)=0,\quad t_{i_0}<t<t_{i_0+1},\quad z_{i_0+1}(t_{i_0};u)=z^{(2)}_{i_0}(t_{i_0};u)+u_{i_0},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
L^*z_N(t;u)=0,\quad t_{N-1}<t<t_N,\quad z_N(t_{N-1};u)=z_{N-1}(t_{N-1};u)+u_{N-1},\\
L^*z_{N+1}(t;u)=0,\quad t_N<t<t_{N+1},\quad z_{N+1}(t_N;u)=z_N(t_N;u)+u_N,\quad \hat{B}_1z_{N+1}(T;u)=0.
\end{gathered}
\tag{6.9}
$$

Using a reasoning similar to the proof of the unique solvability of BVP (2.10), one can show that problem (6.9) is uniquely solvable.

Lemma 6.1. *Finding the minimax estimate of functional $(a,\varphi(s))$ is equivalent to the problem of optimal control of the system described by BVP (6.9) with the cost function*

$$
\begin{aligned}
I(u)=\sum_{i=1,i\neq i_0}^{N+1}\int_{t_{i-1}}^{t_i}&(Q_2^{-1}(t)z_i(t;u),z_i(t;u))_n dt\\
&+\int_{t_{i_0-1}}^{s}(Q_2^{-1}(t)z^{(1)}_{i_0}(t;u),z^{(1)}_{i_0}(t;u))_n dt+\int_s^{t_{i_0}}(Q_2^{-1}(t)z^{(2)}_{i_0}(t;u),z^{(2)}_{i_0}(t;u))_n dt\\
+(Q_0^{-1}\bar{B}_0z_1(0;u),\bar{B}_0z_1(0;u))_m&+(Q_1^{-1}\bar{B}_1z_4(T;u),\bar{B}_1z_4(T;u))_{n-m}+\sum_{i=1}^{N}(\tilde{Q}_i^{-1}u_i,u_i)_n dt\to\inf_{u\in\mathbb{R}^{N\times n}}.
\end{aligned}
\tag{6.10}
$$

Proof. It is easy to see that the following equalities hold

$$
(z_i(t_{i-1};u),\tilde{\varphi}(t_{i-1}))_n-(z_i(t_i;u),\tilde{\varphi}(t_i))_n+\int_{t_{i-1}}^{t_i}(z_i(t;u),\tilde{f}(t))_n dt=0,\tag{6.11}
$$

$$
i=1,\dots,i_0-1,i_0+1,\dots,N+1,
$$

$$
(z^{(1)}_{i_0}(t_{i_0-1};u),\tilde{\varphi}(t_{i_0-1}))_n-(z^{(1)}_{i_0}(s;u),\tilde{\varphi}(s))_n+\int_{t_{i_0-1}}^{s}(z^{(1)}_{i_0}(t;u),\tilde{f}(t))_n dt=0,\tag{6.12}
$$

$$
(z^{(2)}_{i_0}(s;u),\tilde{\varphi}(s))_n-(z^{(2)}_{i_0}(t_{i_0};u),\tilde{\varphi}(t_{i_0}))_n+\int_{s}^{t_{i_0}}(z^{(2)}_{i_0}(t;u),\tilde{f}(t))_n dt=0.\tag{6.13}
$$

Using (6.6), (6.7), and (6.9), we find

$$
\begin{aligned}
(a,\tilde{\varphi}(s))_n-(\widehat{a,\tilde{\varphi}(s)})_n&=(a,\tilde{\varphi}(s))_n-\sum_{i=1}^{N}(u_i,\tilde{y}_i)_n-c\\
=(z^{(1)}_{i_0}(s;u),\tilde{\varphi}(s))_n-(z^{(2)}_{i_0}(s;u),\tilde{\varphi}(s))_n&-\sum_{i=1,i\neq i_0-1,i_0}^{N}(z_{i+1}(t_i;u)-z_i(t_i;u),\tilde{\varphi}(t_i))_n\\
-\Big(z_{i_0+1}(t_{i_0};u)-z^{(2)}_{i_0}(t_{i_0};u),\tilde{\varphi}(t_{i_0})\Big)_n&-\Big(z^{(1)}_{i_0}(t_{i_0-1};u)-z_{i_0-1}(t_{i_0-1};u),\tilde{\varphi}(t_{i_0-1})\Big)_n-\sum_{i=1}^{N}(u_i,\xi_i)_n-c,
\end{aligned}
\tag{6.14}
$$

where the fourth or the third term on the right-hand side of (6.14) should be taken equal to 0 at, respectively, $i_0 = 1$ or $i_0 = N + 1$.

Taking into account the latter, relationships (6.11)–(6.13), and the equalities (see page 7)

$$(z_1(0;u), \tilde{\varphi}(0))_n = \left(\bar{B}_0 z_1(0;u), \tilde{f}_0\right)_m,$$

$$(z_{N+1}(T;u), \tilde{\varphi}(T))_n = \left(\bar{B}_1 z_{N+1}(T;u), \tilde{f}_1\right)_{n-m},$$

we obtain

$$(a, \tilde{\varphi}(s))_n - (\widehat{a, \tilde{\varphi}(s)})_n = (z_1(0;u), \tilde{\varphi}(0))_n - (z_{N+1}(T;u), \tilde{\varphi}(T))_n$$
$$+ \int_0^T (\tilde{z}(t;u), \tilde{f}(t))_n dt - \sum_{i=1}^N (u_i, \xi_i)_n - c$$
$$= \left(\bar{B}_0 z_1(0;u), \tilde{f}_0\right)_m + \int_0^T (\tilde{z}(t;u), \tilde{f}(t))_n dt - \left(\bar{B}_1 z_{N+1}(T;u), \tilde{f}_1\right)_{n-m} - \sum_{i=1}^N (u_i, \xi_i)_n - c, \tag{6.15}$$

where

$$\tilde{z}(t;u) = \begin{cases} z_1(t;u), & t_0 = 0 < t < t_1; \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ z_{i_0-1}(t;u), & t_{i_0-2} < t < t_{i_0-1}; \\ z_{i_0}^{(1)}(t;u), & t_{i_0-1} < t < s; \\ z_{i_0}^{(2)}(t;u), & s < t < t_{i_0}; \\ z_{i_0+1}(t;u), & t_{i_0} < t < t_{i_0+1}; \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ z_{N+1}(t;u), & t_N < t < t_{N+1} = T. \end{cases}$$

Thus,

$$\inf_{c \in \mathbb{R}^1} \sup_{\tilde{F} \in G, \tilde{\xi} \in V} M[(a, \tilde{\varphi}(s))_n - (\widehat{a, \tilde{\varphi}(s)})_n]^2 =$$
$$= \inf_{c \in \mathbb{R}^1} \sup_{\tilde{F} \in G} \left[(\bar{B}_0 z_1(0;u), \tilde{f}_0)_m + \int_0^T (\tilde{z}(t;u), \tilde{f}(t))_n dt - (\bar{B}_1 z_{N+1}(T;u), \tilde{f}_1)_{n-m} - c \right]^2$$
$$+ \sup_{\tilde{\xi} \in V} M \left[\sum_{i=1}^N (u_i, \xi_i)_n \right]^2. \tag{6.16}$$

Calculating the supremum on the right-hand side of (6.16) and taking into consideration (2.3) and (6.4), we find

$$\inf_{c \in \mathbb{R}^1} \sup_{\tilde{F} \in G, \tilde{\xi} \in V} M[(a, \tilde{\varphi}(s))_n - (\widehat{a, \tilde{\varphi}(s)})_n]^2 = I(u),$$

where $I(u)$ is given by (6.10). □

Starting from this lemma and applying the reasoning that led from Lemma 2.1 to Theorems 3.1 and 3.2, we obtain the following results.

Theorem 6.1. *The minimax estimate of expression $(a, \varphi(s))$ has the form*

$$(\widehat{\widehat{a, \varphi(s)}})_n = \sum_{i=1}^N (\hat{u}_i, y_i)_n + \hat{c},$$

where

$$\hat{u}_i = \tilde{Q}_i p_i(t_i), \quad i = 1, \ldots, i_0 - 2, i_0 + 1, \ldots, N, \quad \hat{u}_{i_0-1} = \tilde{Q}_{i_0-1} p_{i_0}^{(1)}(t_{i_0-1}), \quad \hat{u}_{i_0} = \tilde{Q}_{i_0} p_{i_0}^{(2)}(t_{i_0}), \tag{6.17}$$

$$\hat{c} = (\bar{B}_0 z_1(0), f_0^{(0)})_m - (\bar{B}_1 z_{N+1}(T), f_1^{(0)})_{n-m} + \int_0^T (\tilde{z}(t), f^{(0)}(t))_n \, dt,$$

$$\tilde{z}(t) = \begin{cases} z_1(t), & t_0 = 0 < t < t_1; \\ \dots\dots\dots & \dots\dots\dots\dots \\ z_{i_0-1}(t), & t_{i_0-2} < t < t_{i_0-1}; \\ z_{i_0}^{(1)}(t), & t_{i_0-1} < t < s; \\ z_{i_0}^{(2)}(t), & s < t < t_{i_0}; \\ z_{i_0+1}(t), & t_{i_0} < t < t_{i_0+1}; \\ \dots\dots\dots & \dots\dots\dots\dots \\ z_{N+1}(t), & t_N < t < t_{N+1} = T, \end{cases}$$

and vector-functions $p_i(t)$, $z_i(t)$, $i = \overline{1, N+1}, i \neq i_0$, $z_{i_0}^{(1)}(t)$, $p_{i_0}^{(1)}(t)$, $z_{i_0}^{(2)}(t)$, and $p_{i_0}^{(2)}(t)$, are determined from the solution to the equation systems

$$L^* z_1(t) = 0, \quad 0 < t < t_1, \quad \hat{B}_0 z_1(0; u) = 0,$$
$$L^* z_2(t) = 0, \quad t_1 < t < t_2, \quad z_2(t_1) = z_1(t_1) + \tilde{Q}_1 p_1(t_1),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L^* z_{i_0-1}(t) = 0, \quad t_{i_0-2} < t < t_{i_0-1}, \quad z_{i_0-1}(t_{i_0-2}) = z_{i_0-2}(t_{i_0-2}) + \tilde{Q}_{i_0-2} p_{i_0-2}(t_{i_0-2}),$$
$$L^* z_{i_0}^{(1)}(t) = 0, \quad t_{i_0-1} < t < s, \quad z_{i_0}^{(1)}(t_{i_0-1}) = z_{i_0-1}(t_{i_0-1}) + \tilde{Q}_{i_0-1} p_{i_0}^{(1)}(t_{i_0-1}),$$
$$L^* z_{i_0}^{(2)}(t) = 0, \quad s < t < t_{i_0}, \quad z_{i_0}^{(2)}(s) = z_{i_0}^{(1)}(s) - a,$$
$$L^* z_{i_0+1}(t) = 0, \quad t_{i_0} < t < t_{i_0+1}, \quad z_{i_0+1}(t_{i_0}) = z_{i_0}^{(2)}(t_{i_0}) + \tilde{Q}_{i_0} p_{i_0}^{(2)}(t_{i_0}),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L^* z_N(t) = 0, \quad t_{N-1} < t < t_N, \quad z_N(t_{N-1}) = z_{N-1}(t_{N-1}) + \tilde{Q}_{N-1} p_{N-1}(t_{N-1}),$$
$$L^* z_{N+1}(t) = 0, \quad t_N < t < t_{N+1}, \quad z_{N+1}(t_N) = z_N(t_N) + \tilde{Q}_N p_N(t_N), \quad \hat{B}_1 z_{N+1}(T; u) = 0.$$
$$L p_1(t) = Q_2^{-1}(t) z_1(t), \quad 0 < t < t_1, \quad B_0 p_1(0) = Q_0^{-1} \bar{B}_0 z_1(0),$$
$$L p_2(t) = Q_2^{-1}(t) z_2(t), \quad t_1 < t < t_2, \quad p_2(t_1) = p_1(t_1),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L p_{i_0-1}(t) = Q_2^{-1}(t) z_{i_0-1}(t), \quad t_{i_0-2} < t < t_{i_0-1}, \quad p_{i_0-1}(t_{i_0-2}) = p_{i_0-2}(t_{i_0-2}),$$
$$L p_{i_0}^{(1)}(t) = Q_2^{-1}(t) z_{i_0}^{(1)}(t), \quad t_{i_0-1} < t < s, \quad p_{i_0}^{(1)}(t_{i_0-1}) = p_{i_0-1}(t_{i_0-1}),$$
$$L p_{i_0}^{(2)}(t) = Q_2^{-1}(t) z_{i_0}^{(2)}(t), \quad s < t < t_{i_0}, \quad p_{i_0}^{(2)}(s) = p_{i_0}^{(1)}(s),$$
$$L p_{i_0+1}(t) = Q_2^{-1}(t) z_{i_0+1}(t), \quad t_{i_0} < t < t_{i_0+1}, \quad p_{i_0+1}(t_{i_0}) = p_{i_0}^{(2)}(t_{i_0}),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L p_N(t) = Q_2^{-1}(t) z_N, \quad t_{N-1} < t < t_N, \quad p_N(t_{N-1}) = p_{N-1}(t_{N-1}),$$
$$L p_{N+1}(t) = Q_2^{-1}(t) z_{N+1}(t), \; t_N < t < T, \; p_{N+1}(t_N) = p_N(t_N), \; B_1 p_{N+1}(T) = -Q_1^{-1} \bar{B}_1 z_{N+1}(T). \quad (6.18)$$

Here $z_i, p_i \in H^1(t_{i-1}, t_i)^n$, $i = \overline{1, N+1}, i \neq i_0$, $z_{i_0}^{(1)}, p_{i_0}^{(1)} \in H^1(t_{i_0-1}, s)^n$, and $z_{i_0}^{(2)}, p_{i_0}^{(2)} \in H^1(s, t_{i_0})^n$. The minimax estimation error

$$\sigma = (a, p_{i_0}^{(2)}(s))_n^{1/2}. \quad (6.19)$$

System (6.18) is uniquely solvable.

Theorem 6.2. *The following representation is valid*

$$\widehat{\widehat{(a, \varphi(s))_n}} = (a, \hat{\varphi}_{i_0}^{(2)}(s))_n,$$

where vector-functions $\hat\varphi_i(t)$, $i=\overline{1,N+1}, i\neq i_0$, $\hat\varphi_{i_0}^{(1)}(t)$, $\hat\varphi_{i_0}^{(2)}(t)$ are determined from the solution to the equation systems

$$L^*\hat p_1(t)=0,\quad 0<t<t_1,\quad \hat B_0\hat p_1(0;u)=0,$$
$$L^*\hat p_2(t)=0,\quad t_1<t<t_2,\quad \hat p_2(t_1)=\hat p_1(t_1)+\tilde Q_1(\hat\varphi_1(t_1)-y_1),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L^*\hat p_{i_0-1}(t)=0,\quad t_{i_0-2}<t<t_{i_0-1},\quad \hat p_{i_0-1}(t_{i_0-2})=\hat p_{i_0-2}(t_{i_0-2})+\tilde Q_{i_0-2}(\hat\varphi_{i_0-2}(t_{i_0-2})-y_{i_0-2}),$$
$$L^*\hat p_{i_0}^{(1)}(t)=0,\quad t_{i_0-1}<t<s,\quad \hat p_{i_0}^{(1)}(t_{i_0-1})=\hat p_{i_0-1}(t_{i_0-1})+\tilde Q_{i_0-1}(\hat\varphi_{i_0}^{(1)}(t_{i_0-1})-y_{i_0-1}),$$
$$L^*\hat p_{i_0}^{(2)}(t)=0,\quad s<t<t_{i_0},\quad \hat p_{i_0}^{(2)}(s)=\hat p_{i_0}^{(1)}(s),$$
$$L^*\hat p_{i_0+1}(t)=0,\quad t_{i_0}<t<t_{i_0+1},\quad \hat p_{i_0+1}(t_{i_0})=\hat p_{i_0}^{(2)}(t_{i_0})+\tilde Q_{i_0}(\hat\varphi_{i_0}^{(2)}(t_{i_0})-y_{i_0}),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L^*\hat p_N(t)=0,\quad t_{N-1}<t<t_N,\quad \hat p_N(t_{N-1})=\hat p_{N-1}(t_{N-1})+\tilde Q_{N-1}(\hat\varphi_{N-1}(t_{N-1})-y_{N-1}),$$
$$L^*\hat p_{N+1}(t)=0,\quad t_N<t<t_{N+1},\quad \hat p_{N+1}(t_N)=\hat p_N(t_N)+\tilde Q_N(\hat\varphi_N(t_N)-y_N),\quad \hat B_1\hat p_{N+1}(T;u)=0.$$
$$L\hat\varphi_1(t)=Q_2^{-1}(t)\hat p_1(t)+f^{(0)}(t),\quad 0<t<t_1,\quad B_0\hat\varphi_1(0)=Q_0^{-1}\bar B_0\hat p_1(0)+f_0^{(0)},$$
$$L\hat\varphi_2(t)=Q_2^{-1}(t)\hat p_2(t)+f^{(0)}(t),\quad t_1<t<t_2,\quad \hat\varphi_2(t_1)=\hat\varphi_1(t_1),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L\hat\varphi_{i_0-1}(t)=Q_2^{-1}(t)\hat p_{i_0-1}(t)+f^{(0)}(t),\quad t_{i_0-2}<t<t_{i_0-1},\quad \hat\varphi_{i_0-1}(t_{i_0-2})=\hat\varphi_{i_0-2}(t_{i_0-2}),$$
$$L\hat\varphi_{i_0}^{(1)}(t)=Q_2^{-1}(t)\hat p_{i_0}^{(1)}(t)+f^{(0)}(t),\quad t_{i_0-1}<t<s,\quad \hat\varphi_{i_0}^{(1)}(t_{i_0-1})=\hat\varphi_{i_0-1}(t_{i_0-1}),$$
$$L\hat\varphi_{i_0}^{(2)}(t)=Q_2^{-1}(t)\hat p_{i_0}^{(2)}(t)+f^{(0)}(t),\quad s<t<t_{i_0},\quad \hat\varphi_{i_0}^{(2)}(s)=\hat\varphi_{i_0}^{(1)}(s),$$
$$L\hat\varphi_{i_0+1}(t)=Q_2^{-1}(t)\hat p_{i_0+1}(t)+f^{(0)}(t),\quad t_{i_0}<t<t_{i_0+1},\quad \hat\varphi_{i_0+1}(t_{i_0})=\hat\varphi_{i_0}^{(2)}(t_{i_0}),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$L\hat\varphi_N(t)=Q_2^{-1}(t)\hat p_N(t)+f^{(0)}(t),\quad t_{N-1}<t<t_N,\quad \hat\varphi_N(t_{N-1})=\hat\varphi_{N-1}(t_{N-1}),$$
$$L\hat\varphi_{N+1}(t)=Q_2^{-1}(t)\hat p_{N+1}(t)+f^{(0)}(t),\quad t_N<t<T,\quad \hat\varphi_{N+1}(t_N)=\hat\varphi_N(t_N),$$
$$B_1\hat\varphi_{N+1}(T)=-Q_1^{-1}\bar B_1\hat p_{N+1}(T)-f_1^{(0)}.\quad (6.20)$$

Here $\hat p_i=\hat p_i(\cdot;\omega),\hat\varphi_i=\hat\varphi_i(\cdot;\omega)\in H^1(t_{i-1},t_i)^n$, $i=\overline{1,N+1}, i\neq i_0$, $\hat p_{i_0}^{(1)}=\hat p_{i_0}^{(1)}(\cdot;\omega),\hat\varphi_{i_0}^{(1)}=\hat\varphi_{i_0}^{(1)}(\cdot;\omega)\in H^1(t_{i_0-1},s)^n$, $\hat p_{i_0}^{(2)}=p_{i_0}^{(2)}(\cdot;\omega),\hat\varphi_{i_0}^{(2)}=\hat\varphi_{i_0}^{(2)}(\cdot;\omega)\in H^1(s,t_{i_0})^n$, and equalities (6.20) are fulfilled with probability 1. *System (6.20) is uniquely solvable.*

7 Minimax estimation of functionals of solutions to boundary value problems for linear differential equations of order n

In this chapter we propose a method for minimax estimation of parameters of general two-point BVPs for linear ordinary differential equations of order n; their solutions are determined to within functions that are solutions to the corresponding homogeneous problems and exist if the right-hand sides of the equations and boundary conditions entering the problem statement satisfy certain solvability conditions.

7.1 Auxiliary results

If H_0 is a Hilbert space over $\mathbb{C}$ with the inner product $(\cdot,\cdot)_{H_0}$ and norm $\|\cdot\|_{H_0}$, then by $J_{H_0}\in\mathcal{L}(H_0,H_0')$ we will denote an operator called isometric isomorphism; this operator acts from H_0 on its conjugate

space H_0' and is defined by en equality [4] $(v,u)_{H_0} = < v, J_{H_0}u >_{H_0\times H_0'}$ $\forall u, v \in H_0$, where $< x, f >_{H_0\times H_0'}$ $:= f(x)$ for $x \in H_0$, $f \in H_0'$.

Denote by $L^2(a,b)$ the space of functions square integrable on (a,b). For any $n \geq 1$ denote by $W_2^n(a,b)$ the space of functions absolutely continuous on $[a,b]$ together with the derivatives up to order $(n-1)$ for which the derivative of order n that exists almost everywhere on (a,b) belongs to $L^2(a,b)$.

Assume that functions $p_i(t)$ defined on a closed interval $[a,b]$ are such that $p_i^{(n-i)} \in C[a,b]$, $i = \overline{0,n}$, and $p_0(t) \neq 0$ on $[a,b]$. Let functions $\varphi(t)$, belong to the space $W_2^n(a,b)$; define a differential operator L acting in $L^2(a,b)$ by the formula

$$L\varphi(t) = p_0(t)\varphi^{(n)}(t) + p_1(t)\varphi^{(n-1)}(t) + \ldots + p_n(t)\varphi(t).$$

Assume next that there are given a function $f \in L^2(a,b)$ and numbers α_i, $i = \overline{1,m}$. Consider the following BVP: find $\varphi \in W_2^n(a,b)$ that satisfies the equation

$$L\varphi(t) = f(t) \tag{7.1}$$

almost everywhere on (a,b) and the boundary conditions

$$B_i(\varphi) = \alpha_i,\ i = \overline{1,m}, \tag{7.2}$$

where

$$B_i(\varphi) = \sum_{j=0}^{n-1}(\alpha_{i,j}\varphi^{(j)}(a) + \beta_{i,j}\varphi^{(j)}(b)), \quad i = \overline{1,m}, \tag{7.3}$$

is a given system of m linearly independent[5] forms of $2n$ variables

$$\varphi(a), \ldots, \varphi^{(n-1)}(a), \varphi(b), \ldots, \varphi^{(n-1)}(b). \tag{7.4}$$

In order to describe right-hand sides $f \in L^2(a,b)$ and α_i, $i = \overline{1,m}$, for which BVP (7.1), (7.2) is solvable and also to formulate the results obtained in this work, it is necessary to introduce a problem adjoint to (7.1), (7.2). To this end, complement m linearly independent forms $B_i(\varphi)$, $i = \overline{1,m}$, given by (7.4) by some linear forms $S_1(\varphi), \ldots, S_{2n-m}(\varphi)$ to a linearly independent system of $2n$ forms $\{B_1(\varphi), \ldots, B_m(\varphi), S_1(\varphi), \ldots, S_{2n-m}(\varphi)\}$ with respect to the same variables. Denote by L^+ a differential operator which is called formally adjoint to L; this operator is defined on functions from $W_2^n(a,b)$ and act in $L^2(a,b)$ according to

$$L^+\psi(t) = (-1)^n(\overline{p_0(t)}\psi(t))^{(n)} + (-1)^{(n-1)}(\overline{p_1(t)}\psi(t))^{(n-1)} + \ldots + \overline{p_n(t)}\psi(t).$$

Using $B_i(\varphi)$, $i = \overline{1,m}$, and $S_j(\varphi)$, $j = \overline{1,2n-m}$, one can construct systems of linear forms $B_j^+(\psi)$, $j = \overline{1,2n-m}$, and $S_i^+(\psi)$, $i = \overline{1,m}$, with respect to variables $\psi(a), \ldots, \psi^{(n-1)}(a)$, $\psi(b), \ldots, \psi^{(n-1)}(b)$. The constructed forms possess the following properties:
(i) system of forms $\{S_1^+(\psi), \ldots, S_m^+(\psi), B_1^+(\psi), \ldots, B_{2n-m}^+(\psi)\}$ is linearly independent;
(ii) for any $\varphi, \psi \in W_2^n(a,b)$ the Green formula is valid:

$$\int_a^b L\varphi(t)\overline{\psi(t)}\,dt + \sum_{j=1}^{m} B_j(\varphi)\overline{S_j^+(\psi)} = \sum_{j=1}^{2n-m} S_j(\varphi)\overline{B_j^+(\psi)} + \int_a^b \varphi(t)\overline{L^+\psi(t)}\,dt. \tag{7.5}$$

Formulate a BVP: find a function $\psi \in W_2^n(a,b)$ that satisfies the equation

$$L^+\psi(t) = g(t) \tag{7.6}$$

---

[4]This operator exists by the Riesz theorem.

[5]It means that the rank of the matrix composed of the coefficients of these forms equals m.

almost everywhere on (a,b) and the boundary conditions

$$B_j^+(\psi)=\beta_j,\ j=\overline{1,2n-m}, \tag{7.7}$$

where $g\in L^2(a,b)$ is a given function and β_j, $j=\overline{1,2n-m}$ are given numbers. This problem will be called adjoint to BVP (7.1), (7.2).

Problems (7.1), (7.2) and (7.6), (7.7) give rise to operators A_B and A_{B+}^+ defined on functions from $W_2^n(a,b)\subset L^2(a,b)$ according to

$$A_B(\varphi)=\{L\varphi;B_1(\varphi),\dots,B_m(\varphi)\}, \tag{7.8}$$

and

$$A_{B+}^+(\psi)=\{L^+\psi;B_1^+(\psi),\dots,B_{2n-m}^+(\psi)\}; \tag{7.9}$$

the operators act to the spaces $H:=L^2(a,b)\times\mathbb{C}^m$ and $\tilde H:=L^2(a,b)\times\mathbb{C}^{2n-m}$, respectively. From the results proved in [11], it follows that

1) A_B and A_{B+}^+ are Noether operators [6] acting, respectively, from $L^2(a,b)$ to H and $\tilde H$. Kernel $N(A_B)$ of operator A_B has finite dimensionality $n-r$ and coincides with the set $N(A_B)=\{\varphi_0\in C^n[a,b]:L\varphi_0=0$ on (a,b), $B_i(\varphi_0)=0$, $i=\overline{1,m}\}$ of all solutions to homogeneous BVP (7.1), (7.2); kernel $N(A_{B+}^+)$ of operator A_{B+}^+ has dimensionality $m-r$ and coincides with the set $N(A_{B+}^+)=\{\psi_0\in C^n[a,b]:L^+\psi_0=0$ on (a,b), $B_j^+(\psi_0)=0$, $j=\overline{1,2n-m}\}$ of all solutions to homogeneous BVP (7.6), (7.7); number r is the rank of the matrix

$$\begin{pmatrix} B_1(y_1) & B_1(y_2) & \dots & B_1(y_n)\\ B_2(y_1) & B_2(y_2) & \dots & B_2(y_n)\\ \vdots & \vdots & \ddots & \vdots\\ B_m(y_1) & B_m(y_2) & \dots & B_m(y_n)\end{pmatrix}$$

and $y_1(t),\dots,y_n(t)$ is a fundamental system of solutions to homogeneous equation (7.1).

2) BVP (7.1), (7.2) is solvable for given $f\in L^2(a,b)$, $\alpha_i\in\mathbb{C}$, $i=\overline{1,m}$, if and only if the solvability condition

$$\int_a^b f(t)\overline{\psi_0(t)}\,dt+\sum_{i=1}^m\alpha_i\overline{S_i^+(\psi_0)}=0\quad\forall\psi_0\in N(A_{B+}^+) \tag{7.10}$$

holds. If $\varphi(t)$ is a solution to (7.1), (7.2), then $\varphi(t)+\varphi_0(t)$ is also a solution for any $\varphi_0(t)\in N(A_B)$.

3) BVP (7.6), (7.7) is solvable for given $g\in L^2(a,b)$, $\beta_j\in\mathbb{C}$, $j=\overline{1,2n-m}$, if and only if the solvability condition

$$\int_a^b g(t)\overline{\varphi_0(t)}\,dt+\sum_{j=1}^{2n-m}\beta_j\overline{S_j(\varphi_0)}=0\quad\forall\varphi_0\in N(A_B) \tag{7.11}$$

holds. If $\psi(t)$ is a solution to (7.6), (7.7), then $\psi(t)+\psi_0(t)$ is also a solution for any $\psi_0(t)\in N(A_{B+}^+)$.

Let $\varphi_1(t),\dots,\varphi_{n-r}(t)$ and $\psi_1(t),\dots,\psi_{m-r}(t)$ denote in what follows bases of null-spaces $N(A_B)$ and $N(A_{B+}^+)$ of operators A_B and A_{B+}^+, respectively.

7.2 Statement of the estimation problem

An estimation problem can be formulated as follows: to find the optimal (in a certain sense) estimate of the value of the functional

$$l(\varphi)=\int_a^b\overline{l_0(t)}\varphi(t)\,dt \tag{7.12}$$

[6] Recall that A is a Noether operator if the dimensionality of its kernel $N(A)$ is finite and its image $R(A)$ is closed and has a finite codimensionality; then its index $\chi_A=\dim N(A)-\operatorname{codim}R(A)$.

from observations of the form

$$y = C\varphi + \eta \tag{7.13}$$

in the class of estimates

$$\widehat{l(\varphi)} = (y, u)_{H_0} + c, \tag{7.14}$$

linear with respect to observations; here $\varphi(x)$ is a solution to BVP (7.1), u is an element of Hilbert space[7] H_0, $c \in \mathbb{C}$, and $l_0 \in L^2(a,b)$ is a given function. It is assumed that right-hand sides $f(t)$, $\alpha_1, \dots, \alpha_n$ in (7.1), (7.2) and errors $\eta = \eta(\omega)$ in observations (7.13) that are random elements defined on a probability space $(\Omega, \mathcal{B}, P)$ with values in H_0 are not known and it is known only that the element $F := (f(\cdot), \alpha) \in G_0$ and $\eta \in G_1$. Here $\varphi(x)$ is a solution to BVP (7.1), (7.2); $C \in \mathcal{L}(L^2(a,b), H_0)$ is a linear continuous operator such that its restriction on subspace $N(A_B)$ is injective; $\alpha := (\alpha_1, \dots, \alpha_m)^T$ $\in \mathbb{C}^m$ is a vector with components $\alpha_1, \dots, \alpha_m$; G_0 denotes the set of elements

$$\tilde{F} := (\tilde{f}(\cdot), \tilde{\alpha}) = (\tilde{f}(\cdot), (\tilde{\alpha}_1, \dots, \tilde{\alpha}_m)^T) \in L^2(a,b) \times \mathbb{C}^m,$$

satisfying the condition

$$\int_a^b \tilde{f}(t)\overline{\psi_0(t)}\,dt + \sum_{i=1}^m \tilde{\alpha}_i \overline{S_i^+(\psi_0)} = 0 \quad \forall \psi_0 \in N(A_{B^+}^+) \tag{7.15}$$

and the inequality

$$\int_a^b Q(\tilde{f}(t) - f^{(0)}(t))\overline{(\tilde{f}(t) - f^{(0)}(t))}\,dt + (Q_1(\tilde{\alpha} - \alpha^{(0)}), \tilde{\alpha} - \alpha^{(0)})_{\mathbb{C}^m} \le 1, \tag{7.16}$$

in which $(\cdot,\cdot)_{\mathbb{C}^m}$ is the inner product in $\mathbb{C}^m$, element $(f^{(0)}(\cdot), \alpha^{(0)}) = (f^{(0)}(\cdot), (\alpha_1^{(0)}, \dots, \alpha_m^{(0)})^T)$ $\in L^2(a,b) \times \mathbb{C}^m$ and satisfies (7.15), and G_1 is a set of random elements $\tilde{\eta} = \tilde{\eta}(\omega)$ defined on a probability space $(\Omega, \mathcal{B}, P)$ with values in H_0, zero means, and finite second moments $\mathbb{M}\|\tilde{\eta}\|_H^2 < \infty$ satisfying

$$M(Q_0\tilde{\eta}, \tilde{\eta})_{H_0} \le 1, \tag{7.17}$$

where Q and Q_0 are Hermitian operators in $L^2(a,b)$ and H_0, respectively, Q_1 is a Hermitian $m \times m$ matrix for which there exist, respectively, bounded inverse operators Q^{-1} and Q_0^{-1}, and inverse matrix Q_1^{-1}.

Proposition 7.1. *An estimate*

$$\widehat{\widehat{l(\varphi)}} = (y, \hat{u})_{H_0} + \hat{c}$$

for which an element $\hat{u}$ and a constant $\hat{c}$ are determined from the condition

$$\sigma(u,c) := \sup_{\tilde{F} \in G_0, \tilde{\eta} \in G_1} M|l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})}|^2 \to \inf_{u \in H_0, c \in \mathbb{C}} := \sigma^2,$$

where

$$\widehat{l(\tilde{\varphi})} = (\tilde{y}, u)_{H_0} + c, \tag{7.18}$$

$\tilde{y} = C\tilde{\varphi} + \tilde{\eta}$, and $\tilde{\varphi}$ is any solution to BVP (7.1), (7.2) at $f(t) = \tilde{f}(t)$, $\alpha_i = \tilde{\alpha}_i, i = \overline{1,m}$, will be called a minimax estimate of $l(\varphi)$.

The quantity

$$\sigma = \sup_{\tilde{F} \in G_0, \tilde{\eta} \in G_1} \{M|l(\tilde{\varphi}) - \widehat{\widehat{l(\tilde{\varphi})}}|^2\}^{1/2}$$

will be called the minimax estimation error of $l(\varphi)$.

[7] If H_0 is a finite-dimensional space then it is assumed that $\dim H_0 > n - r$.

7.3 Representations for minimax estimates of the values of functionals from solutions and estimation errors

Using the statements formulated in Section 2.1, we arrive at the following results.

Lemma 7.1. *Finding a minimax estimate of the value of functional $l(\varphi)$ is equivalent to the problem of optimal control of the integro-differential equation system*

$$L^+z(t;u) = l_0(t) - C^*J_{H_0}u \quad \text{на} \quad (a,b), \tag{7.19}$$

$$B_j^+(z(\cdot;u)) = 0 \quad (j=1,\dots,2n-m), \tag{7.20}$$

$$\int_a^b Q^{-1}z(t;u)\overline{\psi_i(t)}dt + (Q_1^{-1}\mathbf{S}^+(z(\cdot;u)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i=\overline{1,m-r}, \tag{7.21}$$

with the cost function

$$I(u) = \int_a^b Q^{-1}z(t;u)\overline{z(t;u)}dt + (Q_1^{-1}\mathbf{S}^+(z(\cdot;u)),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m} + (Q_0^{-1}u,u)_{H_0} \to \inf_{u\in U}, \tag{7.22}$$

where

$$U = \{u\in H_0 : \int_a^b \big(l_0(t) - (C^*J_{H_0}u)(t)\big)\overline{\varphi_0(t)}dt = 0 \quad \forall\varphi_0\in N(A_B)\},$$

$C^* : H_0' \to L_2(a,b)$ is the operator adjoint to C defined by the relationship

$$< Cv, w >_{H_0\times H_0'} = \int_a^b v(x)\overline{C^*w(x)}dt \quad \forall v\in L^2(a,b),\ w\in H_0',$$

$\mathbf{l} = (l_1,\dots,l_m)^T$, $\mathbf{S}^+(z(\cdot;u)) := (S_1^+(z(\cdot;u)),\dots,S_m^+(z(\cdot;u)))^T$ *and* $\mathbf{S}^+(\psi_i) := (S_1^+(\psi_i),\dots,S_m^+(\psi_i))^T \in \mathbb{C}^m$ *are vectors with components, respectively,* l_j, $S_j^+(z(\cdot;u))$, *and* $S_j^+(\psi_i)$, $j=\overline{1,m}$.

Proof. Show first that set U is nonempty. Indeed, it is easy to see that U is the meet of $n-r$ hypersurfaces

$$(C\varphi_i,u)_{H_0} = \gamma_i \tag{7.23}$$

in space H_0, where $\gamma_i = \int_a^b \varphi_i(t)l_0(t)\,dt$ and $\varphi_i(t)$, $i=1,\dots,n-r$, is a basis of subspace $N(A_B)$.

Denote by $\mathrm{span}\{C\varphi_1,\dots,C\varphi_{n-r}\}$ a subspace in H_0 spanned over vectors $C\varphi_1,\dots,C\varphi_{n-r}$ and prove that there is one and only one element $u_0\in\mathrm{span}\{C\varphi_1,\dots,C\varphi_{n-r}\}$ that belongs to set U. To this end, representing u_0 as $u_0 = \sum_{j=1}^{n-r}\beta_jC\varphi_j$, where $\beta_j\in\mathbb{C}$, and substituting this into (7.23), we see that u_0 belongs to U if and only if the linear equation system

$$\sum_{j=1}^{n-r}\bar\beta_j(C\varphi_i,C\varphi_j)_{H_0} = \gamma_i,\ i=1,\dots,n-r, \tag{7.24}$$

with respect to unknowns β_j is solvable. Indeed, operator C is injective on $N(A_B)$; therefore, $C\varphi_j$, $j=1,\dots,n-r$, are linearly independent and $\det\{(C\varphi_i,C\varphi_j)_{H_0}\}_{i,j=1}^{n-r}\neq 0$, so that system (7.24) has unique solution $\beta_1,\dots,\beta_{n-r}$. Consequently, the element $u_0=\sum_{j=1}^{n-r}\beta_jC\varphi_j$, belongs, as well as $u_0+u^\perp$ for any $u^\perp\in H_0\ominus\mathrm{span}\{C\varphi_1,\dots,C\varphi_{n-r}\}$, to set U. Thus $U\neq\emptyset$.

Next, let us show that for every fixed $u\in U$, function $z(t;u)$ can be uniquely determined from equalities (7.19)–(7.21). Indeed, the condition $u\in U$ coincides, according to (7.11), with the solvability

condition for problem (7.19)–(7.20). Let $z_0(t;u)\in W_2^n(a,b)$ be a solution to this problem, e.g. a solution that is orthogonal to subspace $N(A^+_{B^+})$; then the function

$$z(t;u):=z_0(t;u)+\sum_{i=1}^{m-r}c_i\psi_i(t), \tag{7.25}$$

also satisfies (7.19)–(7.20) for any $c_i,\in\mathbb{C}^1$, $i=\overline{1,m-r}$. Let us prove that coefficients c_i, $i=\overline{1,m-r}$, can be chosen so that this function would also satisfy (7.21). Substituting expression (7.25) for $z(u)$ into (7.21) we obtain a system of $m-r$ linear algebraic equations with $m-r$ unknowns $c_1,\dots,c_{m-r}$:

$$\sum_{i=1}^{m-r}a_{ij}c_i=b_j(u),\quad j=1,\dots,m-r, \tag{7.26}$$

where

$$a_{ij}=(Q^{-1}\psi_i,\psi_j)_{L^2(a,b)}+(Q_1^{-1}\mathbf{S}^+(\psi_i),\mathbf{S}^+(\psi_j))_{\mathbb{C}^m}, \tag{7.27}$$

$$b_j=-\int_a^b Q^{-1}z_0(t;u)\overline{\psi_j(t)}dt-(Q_1^{-1}\mathbf{S}^+(z_0(\cdot;u)),\mathbf{S}^+(\psi_j))_{\mathbb{C}^m}. \tag{7.28}$$

Show that matrix $[a_{ij}]_{i,j=1}^{m-r}$ of system (7.26) додатно визначена. Indeed, taking into account that Q^{-1} is a Hermitian positive definite operator in $L^2(a,b)$ and Q_1^{-1} is a Hermitian positive definite $m\times m$ matrix, we have

$$\sum_{i=1}^{m-r}\sum_{j=1}^{m-r}a_{ij}\lambda_i\bar\lambda_j=(Q^{-1}\sum_{i=1}^{m-r}\lambda_i\psi_i,\sum_{j=1}^{m-r}\lambda_j\psi_j)_{L^2(a,b)}$$

$$+(Q_1^{-1}(\mathbf{S}^+(\sum_{i=1}^{m-r}\lambda_i\psi_i),\mathbf{S}^+(\sum_{j=1}^{m-r}\lambda_j\psi_j))_{\mathbb{C}^m}\geq c\sum_{i=1}^{m-r}|\lambda_i|^2,\quad c=\text{const}>0,$$

for any λ_i, $i=1,\dots,m-r$, such that $\sum_{i=1}^{m-r}|\lambda_i|^2\neq 0$. The latter implies that matrix $[a_{ij}]_{i,j=1}^{m-r}$ is positive definite. Thus, $\det[a_{ij}]\neq 0$ and system (7.26) has unique solution, $c_1,\dots,c_{m-r}$. Therefore, problem (7.19)–(7.21) is uniquely solvable. Indeed, we have shown that there exists a solution to problem (7.19)–(7.21); let us prove that this solution is unique. Assume that there are two solutions to this problem, $z_1(t)$ and $z_2(t)$. Then

$$L^+z_1(t;u)=l_0(t)-C^*J_{H_0}u\quad\text{on}\quad(a,b), \tag{7.29}$$

$$B_j^+(z_1(\cdot;u))=0\quad j=\overline{1,2n-m}, \tag{7.30}$$

$$\int_a^b Q^{-1}z_1(t;u)\overline{\psi_i(t)}dt+(Q_1^{-1}\mathbf{S}^+(z_1(\cdot;u)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0,\quad i=\overline{1,m-r}, \tag{7.31}$$

$$L^+z_2(t;u)=l_0(t)-C^*J_{H_0}u\quad\text{on}\quad(a,b), \tag{7.32}$$

$$B_j^+(z_2(\cdot;u))=0\quad j=\overline{1,2n-m}, \tag{7.33}$$

$$\int_a^b Q^{-1}z_2(t;u)\overline{\psi_i(t)}dt+(Q_1^{-1}\mathbf{S}^+(z_2(\cdot;u)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0,\quad i=\overline{1,m-r}, \tag{7.34}$$

Subtract (7.32)–(7.34) from equalities (7.29)–(7.31) to obtain

$$L^+(z_1(t;u)-z_2(t;u))=0, \tag{7.35}$$

$$B_j^+(z_1(\cdot;u)-z_2(\cdot;u))=0\quad j=\overline{1,2n-m}, \tag{7.36}$$

$$\int_a^b Q^{-1}(z_1(t;u)-z_2(t;u))\overline{\psi_i(t)}dt+(Q_1^{-1}\mathbf{S}^+(z_1(t;u)-z_2(\cdot;u)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0, \tag{7.37}$$

$$i = \overline{1, m-r}.$$

Set $z(t;u) = z_1(t;u) - z_2(t;u)$; then

$$L^+ z(t;u) = 0, \tag{7.38}$$

$$B_j^+(z(\cdot;u)) = 0 \quad j = \overline{1, 2n-m}, \tag{7.39}$$

$$\int_a^b Q^{-1} z(t;u)\overline{\psi_i(t)}dt + (Q_1^{-1}\mathbf{S}^+(z(\cdot;u)), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m-r}. \tag{7.40}$$

Since $z(t;u)$ solves homogeneous problem (7.19)–(7.20), this function has the form

$$z(t;u) = \sum_{i=1}^{m-r} c_i \psi_i(t). \tag{7.41}$$

Substituting (7.41) into (7.40) we obtain

$$\sum_{i=1}^{m-r} c_i \left(\int_a^b Q^{-1}\psi_i(t)\overline{\psi_j(t)}dt + \sum_{i=1}^{m-r} (Q_1^{-1}\mathbf{S}^+(\psi_i), \mathbf{S}^+(\psi_j)_{\mathbb{C}^m} \right) = 0, \quad j = \overline{1, m-r}, \tag{7.42}$$

or, in line with (7.27),

$$\sum_{i=1}^{m-r} c_i a_{ij} = 0, \quad j = \overline{1, m-r}. \tag{7.43}$$

We see that coefficients c_i satisfy a linear homogeneous algebraic equation system with nonsingular matrix $[a_{ij}]_{i,j=1}^{m-r}$; therefore, this system has only the trivial solution $c_i = 0, i = \overline{1, m-r}$.

By virtue of (7.41), $z(t;u) = z_1(t;u) - z_2(t;u) = 0$ identically, that is, $z_1(t;u) = z_2(t;u)$ which proves the unique solvability of problem (7.19),(7.20).

Next, since any solution $\tilde{\varphi}$ of problem (7.1), (7.2) can be written as $\tilde{\varphi} = \tilde{\varphi}_\perp + \varphi_0$, where $\tilde{\varphi}_0 \in N(A_B)$ and $\tilde{\varphi}_\perp$ is the unique solution to this problem orthogonal to subspace $N(A_B)$, we have

$$\sup_{\tilde{F}\in G_0, \tilde{\eta}\in G_1} M|l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})}|^2 = \sup_{\tilde{F}\in G_0, \tilde{\eta}\in G_1} \sup_{\varphi_0 \in N(A_B)} M|l(\tilde{\varphi}_\perp + \varphi_0) - l(\widehat{\tilde{\varphi}_\perp + \varphi_0})|^2. \tag{7.44}$$

Taking into account (7.12), (7.18), and the fact that

$$\widehat{l(\tilde{\varphi})} = l(\widehat{\tilde{\varphi}_\perp + \varphi_0}) = (C(\tilde{\varphi}_\perp + \varphi_0), u)_{H_0} + (\tilde{\eta}, u)_{H_0} + c$$

$$= < C(\tilde{\varphi}_\perp + \varphi_0), J_{H_0} u >_{H_0 \times H_0'} + (\tilde{\eta}, u)_{H_0} + c$$

$$= \int_a^b (\tilde{\varphi}_\perp(t) + \varphi_0(t))\overline{C^* J_{H_0} u(t)}\, dt + (\tilde{\eta}, u)_{H_0} + c$$

$$= \int_a^b \tilde{\varphi}_\perp(t)\overline{C^* J_{H_0} u(t)}\, dt + \int_a^b \varphi_0(t)\overline{C^* J_{H_0} u(t)}\, dt + (\tilde{\eta}, u)_{H_0} + c,$$

for arbitrary $u \in H_0$, we have

$$l(\tilde{\varphi}) - \widehat{l(\tilde{\varphi})} = \int_a^b \tilde{\varphi}_\perp(t)\overline{(l_0(t) - C^* J_{H_0} u(t))}\, dt + \int_a^b \varphi_0(t)\overline{(l_0(t) - C^* J_{H_0} u(t)}\, dt - (\tilde{\eta}, u)_{H_0} - c.$$

From the latter equality we obtain, taking into consideration the relationship $D\xi = M|\xi - M\xi|^2 = M\xi_1^2 - (M\xi_1)^2 + M\xi_2^2 - (M\xi_2)^2$ that couples dispersion $D\xi$ of a complex-valued random quantity $\xi = \xi_1 + i\xi_2$ and its expectation $M\xi = M\xi_1 + iM\xi_2$,

$$\sup_{\varphi_0 \in N(A_B)} M\left| l(\tilde{\varphi}_\perp(t) + \varphi_0) - l(\widehat{\tilde{\varphi}_\perp(t) + \varphi_0}) \right|^2 = \sup_{\varphi_0 \in N(A_B)} \left| \int_a^b \tilde{\varphi}_\perp(t)(t)\overline{(l_0(t) - C^* J_{H_0} u(t))}\, dt \right.$$

$$+\int_a^b \varphi_0(t)\overline{(l_0(t)-C^*J_{H_0}u(t))}\,dt-c\Bigg|^2+M|(\tilde\eta,u)_{H_0}|^2. \tag{7.45}$$

Since function $\varphi_0(t)$ under the integral sign may be an arbitrary element of space $N(A_B)$, the quantity

$$\sup_{\varphi_0\in N(A_B)} M\left|l(\tilde\varphi_\perp(t)+\varphi_0)-l(\widehat{\tilde\varphi_\perp(t)+\varphi_0})\right|^2$$

will be finite if and only if $u\in U$, i.e. if the second integral on the right-hand side of (7.45) vanishes. Assuming now that $u\in U$ and using (7.19)–(7.21) and (7.5), we obtain

$$\int_a^b \tilde\varphi_\perp(t)\overline{(l_0(t)-C^*J_{H_0}u(t))}\,dt-c=\int_a^b \tilde\varphi_\perp(t)\overline{L^+z(t;u)}\,dt-c$$

$$=\int_a^b L\tilde\varphi_\perp(t)\overline{z(t;u)}\,dt+\sum_{j=1}^m B_j(\tilde\varphi_\perp)\overline{S_j^+(z(\cdot;u))}-c$$

$$=\int_a^b \tilde f(t)\overline{z(t;u)}\,dt+\sum_{j=1}^m \tilde\alpha_j\overline{S_j^+(z(\cdot;u))}-c$$

$$=(\tilde f,z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c.$$

Making use of the latter result together with (7.44) and (7.45), we find

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,\tilde\eta\in G_1} M|l(\tilde\varphi)-\widehat{l(\tilde\varphi)}|^2=$$

$$=\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0}\left|(\tilde f,z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\right|^2+\sup_{\tilde\eta\in G_1}M|(\tilde\eta,u)_{H_0}|^2. \tag{7.46}$$

To calculate the first term on the right-hand side of (7.46) we apply the generalized Cauchy–Bunyakovsky inequality (7.16):

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,}\left|(\tilde f,z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\right|^2$$

$$=\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,}\Big|\overline{(z(\cdot;u),\tilde f-f_0)_{L^2(a,b)}}+\overline{(\mathbf{S}^+(z(\cdot;u)),\tilde\alpha-\alpha^{(0)})_{\mathbb{C}^m}}$$
$$+(f_0,z(\cdot;u))_{L^2(a,b)}+(\alpha^{(0)},\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\Big|^2$$

$$\le\left\{(Q^{-1}z(\cdot;u),z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;u))),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\right\}$$
$$\times\left\{(Q(\tilde f-f^{(0)}),\tilde f-f^{(0)})_{L^2(a,b)}+(Q_1(\tilde\alpha-\alpha^{(0)}),\tilde\alpha-\alpha^{(0)})_{\mathbb{C}^m}\right\}$$

$$\le(Q^{-1}z(\cdot;u),z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;u))),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}. \tag{7.47}$$

Performing direct substitution it is easy to check that inequality (7.47) turns to equality at the element $\tilde F=(\tilde f(\cdot),\tilde\alpha)=\tilde F^{(0)}:=(\tilde f^{(0)}(\cdot),\tilde\alpha^{(0)})=\left(\tilde f^{(0)}(\cdot),(\tilde\alpha_1^{(0)},\ldots,\ \tilde\alpha_m^{(0)})^T\right)\in L^2(a,b)\times\mathbb{C}^m$, where

$$\tilde f^{(0)}(t):=\frac{1}{d}Q^{-1}z(t,u))+f_0(t),$$

$$\tilde\alpha_i^{(0)}:=\frac{1}{d}Q_1^{-1}\mathbf{S}^+(z(\cdot;u))_i+\alpha_i^{(0)},\ i=\overline{1,m},$$

$$d=\Big((Q^{-1}z(\cdot;u),z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}\mathbf{S}^+(z(\cdot;u)),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\Big)^{1/2},$$

and $Q_1^{-1}(\mathbf{S}^+(z(\cdot;u)))_j$ is the jth component of vector $Q_1^{-1}(\mathbf{S}^+(z(\cdot;u)))\in\mathbb{C}^m$. Element $\tilde F^{(0)}\in G_0$ because, obviously, condition (7.16)is fulfilled; in addition,

$$(\tilde f^{(0)},\psi_0)_{L^2(a,b)}+\sum_{i=1}^m\tilde\alpha_i^{(0)}\overline{S_i^+(\psi_0)}=$$

$$=\Big(Q^{-1}z(\cdot;u),z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;u))),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\Big)^{-1/2}$$

$$\times\Big(Q^{-1}z(\cdot;u),\psi_0)_{L^2(a,b)}+\sum_{i=1}^m Q_1^{-1}(\mathbf{S}^+(z(\cdot;u)))_i\overline{S_i^+(\psi_0)}\Big)$$

$$+(f^{(0)},\psi_0)_{L^2(a,b)}+\sum_{i=1}^m\alpha_i^{(0)}\overline{S_i^+(\psi_0)}=0\quad\forall\psi_0\in N(A^+_{B^+})$$

which yields, by virtue of (7.21), the validity of condition (7.15). Therefore,

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0}\Big|(\tilde f,z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\Big|^2$$

$$=\int_a^b Q^{-1}z(t;u)\overline{z(t;u)}dt+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;u))),\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\tag{7.48}$$

at $c=\int_a^b\overline{z(t;u)}f^{(0)}(t)dt+(\alpha^{(0)},\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}$.

In order to calculate the second term in (7.46), note that from the Cauchy–Bunyakovsky inequality (7.17) it follows that

$$|(u,\tilde\eta)_{H_0}|^2\le(Q_0^{-1}u,u)_{H_0}(Q_0\tilde\eta,\tilde\eta)_{H_0},$$

which yields

$$\sup_{\tilde\eta\in G_1}M|(u,\tilde\eta)_{H_0}|^2\le(Q_0^{-1}u,u)_{H_0}.$$

We have

$$M|(u,\tilde\eta^{(0)})_{H_0}|^2=(Q_0^{-1}u,u)_{H_0}$$

where $\tilde\eta^{(0)}=\nu Q_0^{-1}u[(Q_0^{-1}u,u)_{H_0}]^{-1/2}\in G_1$ and ν is a random quantity with $M\nu=0$ and $M|\nu|^2=1$. Therefore

$$\sup_{\tilde\eta\in G_1}M|(u,\tilde\eta)_{H_0}|^2=(Q_0^{-1}u,u)_{H_0}.\tag{7.49}$$

Now the statement of Lemma 2.1 follows directly from relationships (7.46), (7.48), and (7.49). $\square$

Theorem 7.1. *The minimax estimate of $l(\varphi)$ can be represented as*

$$\widehat{\widehat{l(\varphi)}}=(y,\hat u)_{H_0}+\hat c,\tag{7.50}$$

where

$$\hat u=Q_0Cp,\quad\hat c=\int_a^b\overline{z(t)}f^{(0)}(t)dt+\sum_{i=1}^m\overline{S_i^+(z)}\alpha_i^{(0)}\tag{7.51}$$

and functions $p(t)$ and $z(t)$ are determined from the integro-differential equation system

$$L^+z(t)=l_0(t)-C^*J_{H_0}Q_0Cp(t)\quad on\quad(a,b),\tag{7.52}$$

$$B_j^+(z)=0,\quad j=\overline{1,2n-m},\tag{7.53}$$

$$\int_a^b Q^{-1}z(t)\overline{\psi_i(t)}\,dt + (Q_1^{-1}\mathbf{S}^+(z), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m-r}, \tag{7.54}$$

$$Lp(t) = Q^{-1}z(t) \quad on \quad (a,b), \tag{7.55}$$

$$B_j(p) = Q_1^{-1}\mathbf{S}^+(z)_j, \quad j = \overline{1, m}, \tag{7.56}$$

$$\int_a^b (l_0(t) - C^* J_{H_0} Q_0 C p(t))\overline{\varphi_i(t)}dt = 0, \ i = \overline{1, n-r}, \tag{7.57}$$

where $Q_1^{-1}\mathbf{S}^+(z)_j$ denotes the jth component of vector $Q_1^{-1}\mathbf{S}^+(z) \in \mathbb{C}^m$. Problem (7.52)–(7.57) is uniquely solvable. Estimation error σ is determined by $\sigma = l(p)^{1/2}$.

Proof. Let us prove that the solution to the optimal control problem (7.19)–(7.22) can be reduced to the solution of system (7.52)–(7.57). Show first that there exists one and only one element $\hat{u} \in U$ at which the minimum of functional (7.22) is attained: $I(\hat{u}) = \inf_{u \in U} I(u)$.

For an arbitrary $u \in U$ set $u = \bar{u} + v$ where $\bar{u}$ is a fixed element of U and $v = u - \bar{u}$ belongs to the linear subspace $V := \{w \in H_0 : \int_a^b C^* J_{H_0} w(t)\overline{\varphi_0(t)}dt = 0 \quad \forall \varphi_0 \in N(A_B)\}$ of H_0. Let $z(t;\bar{u})$ be the unique solution to the problem

$$L^+ z(t;\bar{u}) = l_0(t) - C^* J_{H_0} \bar{u}(t) \quad \text{on} \quad (a,b), \tag{7.58}$$

$$B_j^+(z(\cdot;\bar{u})) = 0 \quad j = \overline{1, 2n-m}, \tag{7.59}$$

$$\int_a^b Q^{-1}z(t;\bar{u})\overline{\psi_i(t)}dt + (Q_1^{-1}\mathbf{S}^+(z(\cdot;\bar{u})), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m-r}, \tag{7.60}$$

and $\tilde{z}(t;v)$ the unique solution to the problem[8]

$$L^+(t)\tilde{z}(t;v) = -C^* J_{H_0} v \quad \text{on} \quad (a,b), \tag{7.61}$$

$$B_j^+(\tilde{z}(\cdot;v)) = 0 \quad (j = 1, \dots, 2n-m), \tag{7.62}$$

$$\int_a^b Q^{-1}\tilde{z}(t;v)\overline{\psi_i(t)}\,dt + (\tilde{Q}^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m-r}. \tag{7.63}$$

Solution $z(t;u)$ of BVP (7.19)–(7.22) can be represented in the form

$$z(t;u) = z(t;\bar{u}) + \tilde{z}(t;v); \tag{7.64}$$

here if v is an arbitrary element of space V, then $u = \bar{u} + v$ can be an arbitrary element of U. Indeed, adding termwise equalities (7.58)–(7.60) to the corresponding equalities (7.61)–(7.63) we obtain

$$L^+(z(t;\bar{u}) + \tilde{z}(t;v)) = l_0(t) - C^* J_{H_0}(\bar{u}(t) + v(t)) = l_0(t) - C^* J_{H_0} u(t) \quad \text{on} \quad (a,b),$$

$$B_j^+(z(\cdot;\bar{u}) + \tilde{z}(\cdot;v)) = 0 \quad j = \overline{1, 2n-m},$$

$$\int_a^b Q^{-1}(z(t;\bar{u}) + \tilde{z}(t;v))\overline{\psi_i(t)}dt + (Q_1^{-1}(\mathbf{S}^+(z(\cdot;\bar{u}) + \tilde{z}(\cdot;v)), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m-r}.$$

Equating these equalities with the corresponding ones from (7.19)–(7.21) we prove, taking into account the unique solvability of problem (7.19)–(7.21) proved on page 47, the required representation of solution in the form (7.64).

Prove that the solution to BVP (7.61)–(7.63) is continuously dependent on $v \in V$. Consider first operator A_B defined by (7.8); from Green's formula (7.5) and the reasoning similar to that on pp. 90–91 in [11], it follows that the image $\mathcal{D}(A_B^*)$ of operator A_B^* adjoint to A_B, is formed by elements of the

[8] Existence and uniqueness of solution to this problem follows from the condition $v \in V$ and the reasoning that is used in the proof of the unique solvability of problem (7.58)–(7.60).

form $(g(t), (S_1^+(g), \dots, S_m^+(g))^T)$, where g is an arbitrary function from $W_2^n(a,b)$ satisfying the boundary conditions

$$B_j^+(g) = 0 \quad j = 1, \dots, 2n-m, \tag{7.65}$$

and the operator $A_B^* : L^2(a,b) \times \mathbb{C}^m \to L^2(a,b)$ acts according to

$$A_B^*(g(t), (S_1^+(g), \dots, S_m^+(g))^T) = L^+ g. \tag{7.66}$$

Note that A_B^* is a Noether operator because it is adjoint to a Noether one, A_B; therefore,A_B^* is a closed operator and the equations

$$A_B^*(\tilde{z}_0(\cdot;v), (S_1^+(\tilde{z}_0(\cdot;v)), \dots, S_m^+(\tilde{z}_0(\cdot;v)))^T) = L^+\tilde{z}_0(\cdot;v) = -C^* J_{H_0} v \tag{7.67}$$

are normally solvable (that is, $R(A_B^*) = \overline{R(A_B^*)}$). In line with (7.65) and (7.66), the latter equation is equivalent to BVP (7.61)–(7.62). Using this fact and applying Theorem 2.3 from [11][9] to equation (7.67) we obtain that for every $v \in V$ there is a solution $\tilde{z}_0(t;v) \in W_2^n(a,b)$ of problem (7.61)–(7.62) such that

$$\begin{aligned} &\|(\tilde{z}_0(\cdot;v), (S_1^+(\tilde{z}_0(\cdot;v)), \dots, S_m^+(\tilde{z}_0(\cdot;v)))^T)\|_{L^2(a,b)\times\mathbb{C}^m} \\ &\qquad\qquad \le a_1 \|L^+\tilde{z}_0(\cdot;v)\|_{L^2(a,b)} = a_1\|C^* J_{H_0} v\|_{L^2(a,b)} \le a\|v\|_{H_0}, \end{aligned}$$

or, equivalently,

$$\left(\int_a^b |\tilde{z}_0(t;v)|^2 dt + \sum_{i=1}^m |S_i^+(\tilde{z}_0(\cdot;v))|^2 \right)^{1/2} \le a\|v\|_{H_0}, \tag{7.68}$$

where a and a_1 are constants independent of v.

Proceeding similarly to the proof on p. 47, we conclude that the unique solution $\tilde{z}(t;v)$ to BVP (7.61)–(7.63) can be represented in the form

$$\tilde{z}(t;v) = \tilde{z}_0(t;v) + \sum_{i=1}^{m-r} c_i \psi_i(t), \tag{7.69}$$

where the coefficients $c_i = c_i(v) \in \mathbb{C}$ are uniquely determined from the linear algebraic equation system

$$\sum_{i=1}^{m-r} a_{ij} c_i = b_j(v), \quad j = 1, \dots, m-r \tag{7.70}$$

by the formulas

$$c_i(v) = \frac{d_i(v)}{d}, \tag{7.71}$$

where

$$d = \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & a_{i,1} & a_{i+1,1} & \cdots & a_{m-r,1} \\ a_{1,2} & \cdots & a_{i-1,2} & a_{i,2} & a_{i+1,2} & \cdots & a_{m-r,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,m-r} & \cdots & a_{i-1,m-r} & a_{i,m-r} & a_{i+1,m-r} & \cdots & a_{m-r,m-r} \end{vmatrix},$$

[9]Formulate this theorem.

Теорема. *The equation*

$$Ax = y$$

with a closed operator A acting from a Hilbert space E to a Hilbert space F is normally solvable (i.e. $R(A) = \overline{R(A)}$) if and only if for every $y \in R(A)$ there is an $x \in \mathcal{D}(A)$ such that $y = Ax$ and $\|x\| \le k\|Ax\| = k\|y\|$ where $k > 0$ and is independent of $y \in R(A)$.

$$d_i(v) = \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & b_1(v) & a_{i+1,1} & \cdots & a_{m-r,1} \\ a_{1,2} & \cdots & a_{i-1,2} & b_2(v) & a_{i+1,2} & \cdots & a_{m-r,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,m-r} & \cdots & a_{i-1,m-r} & b_{m-r}(v) & a_{i+1,m-r} & \cdots & a_{m-r,m-r} \end{vmatrix}$$

elements $a_{i,j}$, $i, j = 1, \dots, m-r$, of positive definite matrix $[a_{i,j}]$ are determined from (7.27), and

$$b_j(v) = -\int_a^b Q^{-1}\tilde{z}_0(t;v)\overline{\psi_j(t)}dt - (Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v)), \mathbf{S}^+(\psi_j))_{\mathbb{C}^m},\ j = \overline{1, m-r}.$$

Expanding determinant $d_i(v)$ entering the right-hand side of (7.71) in elements of the ith column, we have

$$c_i(v) = \gamma_1^{(i)} b_1(v) + \gamma_2^{(i)} b_2(v) + \cdots + \gamma_{m-r}^{(i)} b_{m-r}(v),\ i = 1, \dots, m-r,$$

here

$$\gamma_j^{(i)} = \frac{A_{ji}}{d}, \quad i, j = 1 \dots, m-r,$$

are constants independent of v, where A_{ji} is the algebraic complement of the element of the ith column that enters the jth row of determinant $d_i(v)$ which is independent of v. Applying the generalized Cauchy–Bunyakovsky inequality to the representation for $c_i(v)$, we obtain

$$\begin{aligned} |c_i(v)| \leq \sum_{j=1}^{m-r} |\gamma_j^{(i)}||b_j(v)| = \sum_{j=1}^{m-r} |\gamma_j^{(i)}| \Big| \int_a^b Q^{-1}\tilde{z}_0(t;v)\overline{\psi_j(t)}\,dt \\ +(Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v)), \mathbf{S}^+(\psi_j))_{\mathbb{C}^m} \Big| \leq \sum_{j=1}^{m-r} |\gamma_j^{(i)}| \Big| \int_a^b Q^{-1}\tilde{z}_0(t;v)\overline{\tilde{z}_0(t;v)}\,dt \\ +(Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v)), \mathbf{S}^+(\tilde{z}_0(\cdot;v)))_{\mathbb{C}^m} \Big|^{1/2} \\ \times \Big| \int_a^b Q^{-1}\psi_j(t)\overline{\psi_j(t)}\,dt + (Q_1^{-1}\mathbf{S}^+(\psi_j), \mathbf{S}^+(\psi_j))_{\mathbb{C}^m} \Big|^{1/2} \\ = A_i \Big| \int_a^b Q^{-1}\tilde{z}_0(t;v)\overline{\tilde{z}_0(t;v)}\,dt + (Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v)), \mathbf{S}^+(\tilde{z}_0(\cdot;v)))_{\mathbb{C}^m} \Big|^{1/2} \end{aligned} \tag{7.72}$$

where

$$A_i = \sum_{j=1}^{m-r} |\gamma_j^{(i)}| \Big| \int_a^b Q^{-1}\psi_j(t)\overline{\psi_j(t)}\,dt + (Q_1^{-1}\mathbf{S}^+(\psi_j), \mathbf{S}^+(\psi_j))_{\mathbb{C}^m} \Big|^{1/2},\ i = \overline{1, m-r},$$

are constants independent of v. Next, taking into consideration the following estimate obtained with the help of the Cauchy–Bunyakovsky inequality and (7.68)

$$\begin{aligned} \Big| \int_a^b Q^{-1}\tilde{z}_0(t;v)\overline{\tilde{z}_0(t;v)}\,dt + (Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v)), \mathbf{S}^+(\tilde{z}_0(\cdot;v)))_{\mathbb{C}^m} \Big|^{1/2} \\ \leq (\|Q^{-1}\tilde{z}_0(\cdot;v)\|_{L^2(a,b)} \|\tilde{z}_0(\cdot;v)\|_{L^2(a,b)} + \|Q_1^{-1}\mathbf{S}^+(\tilde{z}_0(\cdot;v))\|_{\mathbb{C}^m} \|\mathbf{S}^+(\tilde{z}_0(\cdot;v))\|_{\mathbb{C}^m})^{1/2} \\ \leq (\|Q^{-1}\| \|\tilde{z}_0(\cdot;v)\|^2_{L^2(a,b)} + \|Q_1^{-1}\| \|\mathbf{S}^+(\tilde{z}_0(\cdot;v))\|^2_{\mathbb{C}^m})^{1/2} \\ \leq \max\{\|Q^{-1}\|^{1/2}, \|Q_1^{-1}\|^{1/2}\} \left(\int_a^b |\tilde{z}_0(t;v)|^2 dt + \sum_{i=1}^m |S_i^+(\tilde{z}_0(\cdot;v))|^2 \right)^{1/2} \\ \leq a \max\{\|Q^{-1}\|^{1/2}, \|Q_1^{-1}\|^{1/2}\} \|v\|_{H_0}, \end{aligned} \tag{7.73}$$

where constant a enters the right-hand side of (7.68).

Estimates (7.72) and (7.73) yield the inequality

$$|c_i(v)| \leq C_i \|v\|_{H_0}, \; i = 1, \ldots, m-r, \tag{7.74}$$

where $C_i = A_i a \max\{\|Q^{-1}\|^{1/2}, \|Q_1^{-1}\|^{1/2}\}$ are constants independent of v.

Using inequalities (7.74), (7.68) and representation (7.69) of solution $\tilde{z}(t; v)$ to BVP (7.61)–(7.63) we will prove that this solution satisfies the inequality

$$\left(\int_a^b |\tilde{z}(t; v)|^2 dt + \sum_{i=1}^m |S_i^+(\tilde{z}(\cdot; v))|^2 \right) \leq K \|v\|_{H_0}^2, \tag{7.75}$$

where K is a constant independent of v. Taking into notice the inequality $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$ which is valid for any elements a and b from a normed space, we have

$$\int_a^b |\tilde{z}(t; v)|^2 dt + \sum_{i=1}^m |S_i^+(\tilde{z}(\cdot; v))|^2 = \int_a^b |\tilde{z}(t; v)|^2 dt + \|\mathbf{S}^+(z(\cdot; v))\|_{\mathbb{C}^m}^2$$

$$= \int_a^b |\tilde{z}_0(t; v) + \sum_{i=1}^{m-r} c_i(v)\psi_i(t)|^2 dt + \|\mathbf{S}^+(z_0(\cdot; v) + \sum_{i=1}^{m-r} c_i(v)\psi_i(\cdot))\|_{\mathbb{C}^m}^2$$

$$\leq 2 \left(\int_a^b |\tilde{z}_0(t; v)|^2 dt + \sum_{i=1}^{m-r} |c_i(v)|^2 \int_a^b |\psi_i(t)|^2 dt \right)$$

$$+2 \left(\|\mathbf{S}^+(z_0(\cdot; v)\|_{\mathbb{C}^m}^2 + \sum_{i=1}^{m-r} |c_i(v)|^2 \|\mathbf{S}^+(\psi_i)\|_{\mathbb{C}^m}^2 \right)$$

$$= 2 \left(\int_a^b |\tilde{z}_0(t; v)|^2 dt + \sum_{i=1}^m |S_i^+(\tilde{z}_0(\cdot; v))|^2 \right)$$

$$+2 \sum_{i=1}^{m-r} |c_i(v)|^2 \left(\int_a^b |\psi_i(t)|^2 dt + \|\mathbf{S}^+(\psi_i)\|_{\mathbb{C}^m}^2 \right)$$

$$\leq 2a^2 \|v\|_{H_0}^2 + 2 \sum_{i=1}^{m-r} C_i^2 \|v\|_{H_0}^2 \left(\int_a^b |\psi_i(t)|^2 dt + \|\mathbf{S}^+(\psi_i)\|_{\mathbb{C}^m}^2 \right) = K \|v\|_{H_0}^2,$$

where

$$K = 2a^2 + 2 \sum_{i=1}^{m-r} C_i^2 \left(\int_a^b |\psi_i(t)|^2 dt + \|\mathbf{S}^+(\psi_i)\|_{\mathbb{C}^m}^2 \right).$$

Thus, inequality (7.75) is proved.

Reduce now the minimization of functional $I(u)$ on set U to the problem of finding the minimum of the functional

$$I_V(v) := I(\bar{u} + v)$$

on a linear subspace V of space H_0. To this end, make use of representation (7.64) for $z(t; u)$ and write $I(u)$ in the form

$$I(u) = \int_a^b Q^{-1} z(t; u) \overline{z(t; u)} dt + (Q_1^{-1} \mathbf{S}^+(z(\cdot; u)), \mathbf{S}^+(z(\cdot; u))_{\mathbb{C}^m} + (Q_0^{-1} u, u)_{H_0}$$

$$= \int_a^b Q^{-1}(z(t; \bar{u}) + \tilde{z}(t; v)) \overline{(z(t; \bar{u}) + z(t; u))} dt$$

$$+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;\bar u))+\tilde z(\cdot;v)),\mathbf{S}^+(z(\cdot;\bar u)+\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}(\bar u+v),\bar u+v)_{H_0}$$

$$=I(\bar u)+\int_a^b Q^{-1}\tilde z(t;v)\overline{\tilde z(t;v)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}v,v)_{H_0}$$

$$+2\mathrm{Re}\int_a^b Q^{-1}\tilde z(t;v)\overline{z(t;\bar u)}dt+2\mathrm{Re}(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(z(\cdot;\bar u)))_{\mathbb{C}^m}$$

$$+2\mathrm{Re}(Q_0^{-1}v,\bar u)_{H_0}=I(\bar u)+\tilde I(v)+2\mathrm{Re}L(v), \tag{7.76}$$

where, by virtue of estimate (7.75),

$$\tilde I(v)=\int_a^b Q^{-1}\tilde z(t;v)\overline{\tilde z(t;v)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}v,v)_{H_0} \tag{7.77}$$

is a quadratic functional in V associated with the semi-bilinear continuous Hermitian form

$$\pi(v,w)=\int_a^b Q^{-1}\tilde z(t;v)\overline{\tilde z(t;w)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(\tilde z(\cdot;w)))_{\mathbb{C}^m}+(Q_0^{-1}v,w)_{H_0} \tag{7.78}$$

on $V\times V$; the functional satisfies the inequality

$$\tilde I(v)\ge c\|v\|^2_{H_0}\ \forall v\in V,\ \ c=\mathrm{const}, \tag{7.79}$$

and

$$L(v)=\int_a^b Q^{-1}\tilde z(t;v)\overline{z(t;\bar u)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(z(\cdot;\bar u)))_{\mathbb{C}^m}+(Q_0^{-1}v,\bar u)_{H_0} \tag{7.80}$$

is a linear continuous functional in V.

Prove, for example, the continuity of form (7.78), that is, the inequality

$$\pi(v,w)\le c\|v\|_{H_0}\|w\|_{H_0}\quad \forall v,w\in V,\ \ c=\mathrm{const} \tag{7.81}$$

(the continuity of linear functional $L(v)$ can be proved in a similar manner). Using estimate (7.75) and the Cauchy–Bunyakovsky inequality, we have

$$|\pi(v,w)|\le\left(\int_a^b Q^{-1}\tilde z(t;v)\overline{\tilde z(t;v)}dt\right)^{1/2}\left(\int_a^b Q^{-1}\tilde z(t;w)\overline{\tilde z(t;w)}dt\right)^{1/2}$$

$$+\left(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(\tilde z(\cdot;v))\right)^{1/2}_{\mathbb{C}^m}\left(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;w)),\mathbf{S}^+(\tilde z(\cdot;w))\right)^{1/2}_{\mathbb{C}^m}$$

$$+(Q_0^{-1}v,v)^{1/2}_{H_0}(Q_0^{-1}w,w)^{1/2}_{H_0}$$

$$\le\left(\int_a^b Q^{-1}\tilde z(t;v)\overline{\tilde z(t;v)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v)),\mathbf{S}^+(\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}v,v)_{H_0}\right)^{1/2}$$

$$\times\left(\int_a^b Q^{-1}\tilde z(t;w)\overline{\tilde z(t;w)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;w)),\mathbf{S}^+(\tilde z(\cdot;w)))_{\mathbb{C}^m}+(Q_0^{-1}w,w)_{H_0}\right)^{1/2}$$

$$\le\left\{\left(\int_a^b|Q^{-1}\tilde z(t;v)|^2dt\right)^{1/2}\left(\int_a^b|\tilde z(t;v)|^2dt\right)^{1/2}\right.$$

$$\left.+\|Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;v))\|_{\mathbb{C}^m}\|\mathbf{S}^+(\tilde z(\cdot;v))\|_{\mathbb{C}^m}+\|Q_0^{-1}v\|_{H_0}\|v\|_{H_0}\right\}^{1/2}$$

$$\times \{\left(\int_a^b |Q^{-1}\tilde z(t;w)|^2dt\right)^{1/2}\left(\int_a^b |\tilde z(t;w)|^2dt\right)^{1/2}$$

$$+\|Q_1^{-1}\mathbf{S}^+(\tilde z(\cdot;w))\|_{\mathbb{C}^m}\|\mathbf{S}^+(\tilde z(\cdot;w))\|_{\mathbb{C}^m}+\|Q_0^{-1}w\|_{H_0}\|w\|_{H_0}\Bigg\}^{1/2}$$

$$\le \max\{\|Q^{-1}\|,\|Q_1^{-1}\|,\|Q_0^{-1}\|\}\left\{\int_a^b |\tilde z(t;v)|^2dt+\|\mathbf{S}^+(\tilde z(\cdot;v))\|^2_{\mathbb{C}^m}+\|v\|^2_{H_0}\right\}^{1/2}$$

$$\times\left\{\int_a^b |\tilde z(t;w)|^2dt+\|\mathbf{S}^+(\tilde z(\cdot;w))\|^2_{\mathbb{C}^m}+\|w\|^2_{H_0}\right\}^{1/2}$$

$$\le \max\{\|Q^{-1}\|,\|Q_1^{-1}\|,\|Q_0^{-1}\|\}(K\|v\|^2_{H_0}+\|v\|^2_{H_0})^{1/2}(K\|w\|^2_{H_0}+\|w\|^2_{H_0})^{1/2}$$

$$\le c\|v\|_{H_0}\|w\|_{H_0},$$

where

$$c=\max\{\|Q^{-1}\|,\|Q_1^{-1}\|,\|Q_0^{-1}\|\}(K+1).$$

Thus, we have proved inequality (7.81) and consequently the continuity of form (7.78).

Taking into consideration the continuity of (7.78) and Remark 1.1 to Theorem 1.1 from [3] we see that there exists the unique element $\hat v\in V$ (dependent on $\bar u$) such that

$$I_V(\hat v)=\inf_{v\in V}I_V(v),=\inf_{v\in V}I(\bar u+v)=\inf_{u-\bar u\in V}I(u)=\inf_{u\in\bar u+V}I(u)=\inf_{u\in U}I(u).$$

Setting $\hat u=\bar u+\hat v$ and using the equality

$$I_V(\hat v)=I(\bar u+\hat v)=I(\hat u)$$

we conclude that there exists one and only one element $\hat u=\bar u+\hat v$, $\hat u\in U$, such that functional $I(u)$ attaints the minimum at $u\in U$. Therefore, for any $\tau\in R$ and $v\in V$,

$$\frac{d}{d\tau}I(\hat u+\tau v)\Big|_{\tau=0}=0\quad\text{and}\quad\frac{d}{d\tau}I(\hat u+i\tau v)\Big|_{\tau=0}=0,\tag{7.82}$$

where $i=\sqrt{-1}$. Since $z(t;\hat u+\tau v)=z(t;\hat u)+\tau\tilde z(t;v)$, where $\tilde z(t;v)$ is the unique solution to BVP (7.19)–(7.21) at $u=v$ and $l_0=0$, from the first relationship (7.82) we obtain

$$0=\frac{1}{2}\frac{d}{d\tau}I(\hat u+\tau v)|_{\tau=0}$$

$$=\lim_{\tau\to0}\frac{1}{2\tau}\Big\{\Big[(Q^{-1}z(\cdot;\hat u+\tau v),z(\cdot;\hat u+\tau v))_{L^2(a,b)}-(Q^{-1}z(\cdot;\hat u),z(\cdot;\hat u))_{L^2(a,b)}\Big]$$

$$+\Big[(Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat u+\tau v))),\mathbf{S}^+(z(\cdot;\hat u+\tau v)))_{\mathbb{C}^m}-(Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat u))),\mathbf{S}^+(z(\cdot;\hat u)))_{\mathbb{C}^m}\Big]$$

$$+\Big[Q_0^{-1}(\hat u+\tau v),\hat u+\tau v)_{H_0}-(Q_0^{-1}\hat u,\hat u)_{H_0}\Big]\Big\}$$

$$=\mathrm{Re}\{(Q^{-1}z(\cdot;\hat u),\tilde z(\cdot;v))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat u))),\mathbf{S}^+(\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}\hat u,v)_{H_0}\}.$$

Similarly, taking into account that $z(t;\hat u+i\tau v)=z(t;\hat u)+i\tau\tilde z(t;v)$,

$$0=\frac{1}{2}\frac{d}{d\tau}I(\hat u+i\tau v)|_{\tau=0}$$

$$=\mathrm{Im}\{(Q^{-1}z(\cdot;\hat u),\tilde z(\cdot;v))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat u))),\mathbf{S}^+(\tilde z(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}\hat u,v)_{H_0}\},$$

which yields

$$(Q^{-1}z(\cdot;\hat{u}),\tilde{z}(\cdot;v))_{L^2(a,b)}+\sum_{j=1}^{m}Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat{u})))_j\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}+(Q_0^{-1}\hat{u},v)_{H_0}=0. \tag{7.83}$$

Let $p(t)$ be a solution to the BVP[10]

$$Lp(t)=Q^{-1}z(t;\hat{u})\quad\text{on}\quad(a,b), \tag{7.84}$$

$$B_j(p)=Q_1^{-1}(\mathbf{S}^+(z(\cdot,\hat{u})))_j,\quad j=\overline{1,m}. \tag{7.85}$$

Then the sum of the first two terms on the left-hand side of (7.83) can be written, in view of Green's formula (7.5), in the form

$$(Q^{-1}z(\cdot;\hat{u}),\tilde{z}(\cdot;v))_{L^2(a,b)}+\sum_{j=1}^{m}Q_1^{-1}(\mathbf{S}^+(z(\cdot;\hat{u})))_j\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}$$

$$=\int_a^b Lp(t)\overline{\tilde{z}(t;v)}\,dt+\sum_{j=1}^{m}B_j(p)\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}=\int_a^b p(t)\overline{L^+\tilde{z}(t;v)}\,dt$$

$$=-\int_a^b p(t)\overline{C^*J_{H_0}v(t)}\,dt=-<Cp,J_{H_0}v>_{H_0\times H_0'}=-(Cp,v)_{H_0}.$$

From the latter equality and formula (7.83), it follows that for any $v\in V$,

$$(Q_0^{-1}\hat{u}-Cp,v)_{H_0}=0. \tag{7.86}$$

Let us show that in the set of solutions to problem (7.84), (7.85) there is only one, $p(t)$, for which

$$Q_0^{-1}\hat{u}-Cp\in V. \tag{7.87}$$

Indeed, condition (7.87) means that for any $1\le i\le n-r$ the equalities

$$\int_a^b\varphi_i(t)\overline{C^*J_{H_0}(Q_0^{-1}\hat{u}-Cp)(t)}\,dt=0 \tag{7.88}$$

hold. Since general solution $p(t)$ to BVP (7.84), (7.85) has the form

$$p(t)=\tilde{p}(t)+\sum_{j=1}^{n-r}a_j\varphi_j(t),$$

where $\tilde{p}(t)$ is a particular solution to this problem and $a_j\in\mathbb{C}$ $(j=\overline{1,n-r})$ are arbitrary numbers, we conclude that in line with (7.88), function $p(t)$ satisfies condition (7.87) if $(a_1,\dots,a_{n-r})^T$ is a solution to the uniquely solvable linear algebraic equation system

$$\sum_{i=1}^{n-r}a_i(C\varphi_i,C\varphi_j)_{H_0}=(Q_0^{-1}\hat{u}-C\tilde{p},C\varphi_j)_{H_0},\quad j=\overline{1,n-r},$$

where matrix $[(C\varphi_i,C\varphi_j)_{H_0}]_{i,j=1}^{n-r}$ has a non-zero determinant bacause it is the Gram matrix of the system of linearly independent elements $C\varphi_1,\dots,C\varphi_{n-r}$. It is easy to see that the unique solvability of this system yields the existence of the unique function $p(t)$ that satisfies condition (7.87) and equations (7.84) and (7.85).

[10]Relationship (7.54) coincides with the solvability condition for this problem by virtue of (7.10).

Setting in (7.86) $v = Q_0^{-1}\hat{u} - Cp$ we have $Q_0^{-1}\hat{u} - Cp = 0$, so that $\hat{u} = Q_0 Cp$. Substituting the latter into $\int_a^b (l_0(t) - C^* J_{H_0}\hat{u}(t))\overline{\varphi_0(t)}dt = 0$ and denoting $z(t;\hat{u}) =: z(t)$, we see that $z(t)$ and $p(t)$ satisfy system (7.52)–(7.57); the unique solvability of this system follows from the uniqueness of element $\hat{u}$.

Prove now that $\sigma(\varphi) \le \sigma(\hat{u}, \hat{c}) = l(p)^{1/2}$. Substituting $\hat{u} = Q_0 Cp$ into $I(\hat{u})$ and taking into account the designation $z(t) = z(t;\hat{u})$, we obtain

$$I(\hat{u}) = \int_a^b Q^{-1} z(t)\overline{z(t)}\,dt + (Q_1^{-1}\mathbf{S}^+(z), \mathbf{S}^+(z))_{C^m}$$

$$+(Cp, Q_0 Cp)_{H_0} = \int_a^b Lp(t)\overline{z(t)}\,dt$$

$$+\sum_{j=1}^{m}(Q_1^{-1}\mathbf{S}^+(z))_j(\overline{S_j^+(z)}) + (Cp, Q_0 Cp)_{H_0}$$

$$= \int_a^b Lp(t)\overline{z(t)}\,dt + \sum_{j=1}^{m} B_j(p)\overline{S_j^+(z)} + (Cp, Q_0 Cp)_{H_0}$$

$$= \int_a^b p(t)\overline{L^+ z(t)}\,dt + \sum_{j=1}^{2n-m} S_j(p)\overline{B_j^+(z)} + (Cp, Q_0 Cp)_{H_0}. \tag{7.89}$$

For the first term in (7.89) we have

$$\int_a^b p(t)\overline{L^+ z(t)}\,dt = \int_a^b p(t)\overline{l_0(t)}\,dt - \int_a^b p(t)(\overline{C^* J_{H_0} Q_0 Cp)(t)})\,dt$$

$$= \int_a^b p(t)\overline{l_0(t)}\,dt - < Cp, J_{H_0} Q_0 Cp >_{H_0 \times H_0'} .$$

The later equality together with (7.89) yield $I(\hat{u}) = l(p)$. The theorem is proved. □

Theorem 7.2. *The minimax estimate of $l(\varphi)$ has the form*

$$\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi}),$$

where function $\hat{\varphi}$ is determined from the solution to the problem

$$L^+\hat{p}(t) = C^* J_{H_0} Q_0 (y - C\hat{\varphi})(t) \quad on \quad (a, b), \tag{7.90}$$

$$B_j^+(\hat{p}) = 0, \quad j = \overline{1, 2n - m}, \tag{7.91}$$

$$\int_a^b Q^{-1}\hat{p}(t)\overline{\psi_i(t)}\,dt + (Q_1^{-1}\mathbf{S}^+(\hat{p}), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1, m - r}, \tag{7.92}$$

$$L\hat{\varphi}(t) = Q^{-1}\hat{p}(t) + f^{(0)}(t) \quad on \quad (a, b), \tag{7.93}$$

$$B_j(\hat{\varphi}) = Q_1^{-1}\mathbf{S}^+(\hat{p})_j + \alpha_j^{(0)}, \quad j = \overline{1, m}, \tag{7.94}$$

$$\int_a^b C^* J_{H_0} Q_0 \left(y - C\hat{\varphi}\right)(t)\overline{\varphi_i(t)}\,dt = 0, \quad i = \overline{1, n - r}. \tag{7.95}$$

Problem (7.90)–(7.95) is uniquely solvable.

Proof. Consider the problem of optimal control of the equation system

$$L^{+}\hat{p}(t;u) = -(C^{*}J_{H_0}u)(t) + (C^{*}J_{H_0}Q_0 y)(t) \quad \text{on} \quad (a,b), \tag{7.96}$$

$$B_j^{+}(\hat{p}(\cdot;u)) = 0 \quad (j = 1,\dots,m), \tag{7.97}$$

$$\int_a^b Q^{-1}\hat{p}(t;u)\overline{\psi_i(t)}dt + (Q_1^{-1}\mathbf{S}^{+}(\hat{p}(\cdot;u)), \mathbf{S}^{+}(\psi_i))_{\mathbb{C}^m} = 0, \quad i = \overline{1,m-r}, \tag{7.98}$$

with the cost function

$$I(u) = \int_a^b Q^{-1}(\hat{p}(t;u) + Qf^{(0)}(t))\overline{(\hat{p}(t;u) + Qf^{(0)}(t))}dt$$
$$+ (Q_1^{-1}(\mathbf{S}^{+}(\hat{p}(\cdot;u)) + Q_1\alpha^{(0)}), \mathbf{S}^{+}(\hat{p}(\cdot;u)) + Q_1\alpha^{(0)})_{\mathbb{C}^m} + (Q_0^{-1}u,u)_{H_0} \to \inf_{u\in\tilde{U}}, \tag{7.99}$$

where $\tilde{U} = \{u \in H_0 : \int_a^b (C^{*}J_{H_0}Q_0 y)(t) - C^{*}J_{H_0}u(t))\overline{\varphi_0(t)}dt = 0\}$ for arbitrary solutions $\varphi_0(t)$ of homogeneous BVP (7.1), (7.2).

The form of functional $I(u)$ and the reasoning contained in the proof of Theorem 2.1 suggest the existence of the unique element $\hat{u} \in \tilde{U}$ such that

$$I(\hat{u}) = \inf_{u\in\tilde{U}} I(u).$$

Next, finding $\hat{\varphi}(t)$ as the unique solution to the BVP

$$L\hat{\varphi}(t) = Q^{-1}\hat{p}(t;\hat{u}) + f^{(0)}(t) \quad \text{on} \quad (a,b),$$

$$B_j(\hat{\varphi}) = Q_1^{-1}\mathbf{S}^{+}(\hat{p}(\cdot;\hat{u})_j + \alpha_j^{(0)}, \quad j = \overline{1,m},$$

$$\int_a^b C^{*}J_{H_0}Q_0\,(y - C\hat{\varphi})\,(t)\overline{\varphi_i(t)}\,dt = 0, \quad i = \overline{1,n-r},$$

and denoting $\hat{p}(t) = \hat{p}(t;\hat{u})$, we conclude, repeating virtually the proof of Theorem 2.1, that problem (7.90)–(7.95) is uniquely solvable.

Now let us prove the representation $\widehat{\widehat{l(\varphi)}} = l(\hat{\varphi})$. Substituting expression (7.51) for $\hat{u}$ and $\hat{c}$ into (7.50) and taking into consideration relationships (7.90)–(7.92), we obtain

$$\widehat{\widehat{l(\varphi)}} = (y,\hat{u})_{H_0} + \hat{c} = (y, Q_0 C p)_{H_0} + \hat{c} = \overline{(Cp, Q_0 y)_{H_0}} + \hat{c}$$
$$= \overline{< Cp, J_{H_0}Q_0 y >_{H_0\times H_0'}} = \overline{(p, C^{*}J_{H_0}Q_0 y)_{L^2(a,b)}} + \hat{c}$$
$$= \overline{\int_a^b p(t)\overline{C^{*}J_{H_0}Q_0 y(t)}\,dt} + \hat{c} = \overline{\int_a^b p(t)\overline{L^{+}\hat{p}(t)}\,dt} + \int_a^b \overline{z(t)}f^{(0)}(t)dt$$
$$+ \sum_{j=1}^{m} \overline{S_j^{+}(z)}\alpha_j^{(0)} + \overline{\int_a^b p(t)\overline{C^{*}J_{H_0}Q_0 C\hat{\varphi}(t)}\,dt}. \tag{7.100}$$

Transform the sum of the first three terms on the right-hand side of this equality using Green's formula (7.5) and equalities (7.52)–(7.57) and (7.93)–(7.95). We have

$$\overline{\int_a^b p(t)\overline{L^{+}\hat{p}(t)}\,dt} + \int_a^b \overline{z(t)}f^{(0)}(t)dt + \sum_{j=1}^{m} \overline{S_j^{+}(z)}\alpha_j^{(0)}$$

$$= \overline{\int_a^b Lp(t)\overline{\hat{p}(t)}\,dt} + \sum_{j=1}^{m} \overline{B_j(p)\overline{S_j^{+}(\hat{p}(t))}}$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\overline{\int_a^b Q^{-1}z(t)\overline{\hat{p}(t)}\,dt}+\sum_{j=1}^{m}\overline{Q_1^{-1}\mathbf{S}^+(z)_j\overline{S_j^+(\hat{p})}}$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\overline{\int_a^b Q^{-1}z(t)\overline{\hat{p}(t)}\,dt}+\overline{(Q_1^{-1}\mathbf{S}^+(z),\mathbf{S}^+(\hat{p}))_{\mathbb{C}^m}}$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\overline{\int_a^b z(t)\overline{Q^{-1}\hat{p}(t)}\,dt}+\overline{(\mathbf{S}^+(z),Q_1^{-1}\mathbf{S}^+(\hat{p}))_{\mathbb{C}^m}}$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\int_a^b \overline{z(t)}Q^{-1}\hat{p}(t)\,dt+\sum_{j=1}^{m}\overline{S_j^+(z)\overline{(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j}}$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\int_a^b \overline{z(t)}Q^{-1}\hat{p}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j$$

$$+\int_a^b \overline{z(t)}f^{(0)}(t)dt+\sum_{j=1}^{m}\overline{S_j^+(z)}\alpha_j^{(0)}$$

$$=\int_a^b \overline{z(t)}(Q^{-1}\hat{p}(t)+f^{(0)}(t))dt+\sum_{j=1}^{m}\overline{S_j^+(z)}(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j+\alpha_j^{(0)})$$

$$=\int_a^b L\hat{\varphi}(t)\overline{z(t)}dt+\sum_{j=1}^{m}B_j(\hat{\varphi})\overline{S_j^+(z)}=\int_a^b \hat{\varphi}(t)\overline{L^+z(t)}dt$$

$$=l(\hat{\varphi})-\int_a^b \hat{\varphi}(t)\overline{C^*J_{H_0}Q_0Cp(t)}dt=l(\hat{\varphi})-<C\hat{\varphi},J_{H_0}Q_0Cp>_{H_0\times H_0'}$$

$$=l(\hat{\varphi})-(C\hat{\varphi},Q_0Cp)_{H_0}=l(\hat{\varphi})-(Q_0C\hat{\varphi},Cp)_{H_0}=l(\hat{\varphi})-\overline{(Cp,Q_0C\hat{\varphi})_{H_0}}$$
$$=l(\hat{\varphi})-\overline{<Cp,J_{H_0}Q_0C\hat{\varphi}>_{H_0\times H_0'}}=l(\hat{\varphi})-\overline{(p,C^*J_{H_0}Q_0C\hat{\varphi})_{L^2(a,b)}}$$
$$=l(\hat{\varphi})-\overline{\int_a^b p(t)\overline{C^*J_{H_0}Q_0C\hat{\varphi}(t)}dt}.$$

The latter equality together with (7.100) prove the sought-for representation. ☐

Corollary 7.1. *Function $\hat{\varphi}(t)$ can be taken as an estimate of solution $\varphi(t)$ to BVP* (7.1), (7.2) *which is observed.*

Proposition 7.1. *Statements similar to Theorems* 7.1 *and* 7.2 *can be obtained in the case when errors* η *in observations* (7.13) *are deterministic elements with values in space* H_0.

As an example, consider the case $H_0 = (L^2(a,b))^N$ and the operator $C : L^2(a,b) \to H_0$ in observations (7.13) is defined by the equality

$$C\varphi(t) = \left(\int_a^b K_1(t,\xi)\varphi(\xi)\,d\xi, \dots, \int_a^b K_N(t,\xi)\varphi(\xi)\,d\xi \right)^T,$$

where kernels $K_j \in L^2(a,b) \times L^2(a,b)$ of the integral operators

$$C_j\varphi(t) := \int_a^b K_j(t,\xi)\varphi(\xi)\,d\xi, \quad j = \overline{1,N},$$

are assumed to be such that the vector-functions

$$C\varphi_i(t) = \left(\int_a^b K_1(t,\xi)\varphi_i(\xi)\,d\xi, \dots, \int_a^b K_N(t,\xi)\varphi_i(\xi)\,d\xi \right)^T \in \left(L^2(a,b)\right)^N,$$

$i = \overline{1, n-r}$, are linearly independent. Observations (7.13) take the form

$$y_i(t) = \int_a^b K_i(t,\xi)\varphi(\xi)\,d\xi + \eta_i(t), \quad i = \overline{1,N},$$

where $\eta(t) := (\eta_1(t), \dots, \eta_N(t)) \in G_1$ is a random vector process with components $\eta_j(t)$ which are random processes with zero expectations and finite second moments and operator Q_0 in condition (7.17) that specifies set G_1 is identified with an $N \times N$ matrix that has elements $q_{ij}^{(0)} \in C[a,b]$. Consequently, this condition takes the form $\int_a^b \mathrm{Sp}(Q_0(t)\tilde{R}(t,t))\,dt \le 1$, in which $\tilde{R}(t,s) = [\tilde{b}_{i,j}(t,s)]_{i,j=1}^N$ denotes the unknown correlation matrix of the vector process $\tilde{\eta} = (\tilde{\eta}_1(t), \dots, \tilde{\eta}_N(t))$ with the elements $\tilde{b}_{i,j}(t,s) = \mathbf{M}\tilde{\eta}_i(t)\overline{\tilde{\eta}_j(s)}$ and $\mathrm{Sp}(Q_0(t)\tilde{R}(t,t))$ denotes the trace of matrix $Q_0(t)\tilde{R}(t,t)$.

It is easy to see now that the operator $C^* : (L^2(a,b))^N \to L^2(a,b)$ adjoint to C is given by the formula

$$(C^*\psi)(t) = (C^*(\psi_1(\cdot), \dots, \psi_N(\cdot))^T)(t) = \sum_{j=1}^N \int_a^b \overline{K_j(\xi,t)}\psi_j(\xi)\,d\xi,$$

and minimax estimate $\widehat{\widehat{l(\varphi)}}$ has the form

$$\widehat{\widehat{l(\varphi)}} = \sum_{i=1}^N \int_a^b y_i(t)\overline{\hat{u}_i(t)}\,dt + \hat{c},$$

where

$$\hat{u}_i(t) = \sum_{j=1}^N q_{ij}^{(0)}(t) \int_a^b K_j(t,\xi)p(\xi)\,d\xi, \quad i = \overline{1,N}.$$

Equalities (7.52) and (7.57) become

$$L^+ z(t) = l_0(t) - \int_a^b \tilde{K}(t,\xi)p(\xi)\,d\xi \quad \text{on} \quad (a,b), \tag{2.35'}$$

and

$$\int_a^b \left(l_0(t) - \tilde{K}(t,\xi)p(\xi)\,d\xi \right) \overline{\varphi_{i'}(t)}dt = 0, \; i' = \overline{1, n-r}, \tag{2.40'}$$

where

$$\tilde{K}(t,\xi) = \sum_{i=1}^N \sum_{j=1}^N \int_a^b \overline{K_i(t',t)} q_{ij}^{(0)}(t') K_j(t',\xi)\,dt'.$$

8 Minimax estimation of functionals from right-hand sides of equations that enter the problem statements. Representations for minimax estimates and estimation errors

An estimation problem in question can be formulated as follows: to find the optimal (in a certain sense) estimate of the value of the functional

$$l(F) = \int_a^b \overline{l_0(t)} f(t)\,dt + \sum_{j=1}^m \overline{l_j}\alpha_j, \tag{8.1}$$

from observations of the form

$$y = C\varphi + \eta \tag{8.2}$$

in the class of estimates

$$\widehat{l(F)} = (y, u)_{H_0} + c, \tag{8.3}$$

linear with respect to observations; here u belongs to Hilbert space H_0, $c \in \mathbb{C}$, $l_0 \in L^2(a,b)$ is a given function, and $l_j \in \mathbb{C}$, $j = \overline{1,m}$ are given numbers; the assumption is that $F := (f(\cdot), \alpha) = (f(\cdot), (\alpha_1, \ldots, \alpha_n)) \in G_0$ and the errors $\eta = \eta(\omega)$ in observations (8.2) belong to G_1, where sets G_0 and G_1 are specified, respectively, by (7.15), (7.16), and (7.17).

Proposition 8.1. *An estimate* $\widehat{\widehat{l(F)}} = (y, \hat{u})_{H_0} + \hat{c}$ *for which an element* $\hat{u}$ *and a constant* $\hat{c}$ *are determined from the condition*

$$\sup_{\tilde{F} \in G_0, \tilde{\eta} \in G_1} M|l(\tilde{F}) - \widehat{l(F)}|^2 \to \inf_{u \in H_0,\, c \in \mathbb{C}},$$

where $\widehat{l(\tilde{F})} = (\tilde{y}, u)_{H_0} + c$, $\tilde{y} = C\tilde{\varphi} + \tilde{\eta}$, *and* $\tilde{\varphi}$ *is any solution to BVP (7.1), (7.2) at* $f(t) = \tilde{f}(t)$ *and* $\alpha_i = \tilde{\alpha}_i$, $i = \overline{1,m}$, *will be called a minimax estimate of* $l(F)$.

The quantity $\sigma := \sup_{\tilde{F} \in G_0, \tilde{\eta} \in G_1} \{M|l(\tilde{F}) - \widehat{\widehat{l(\tilde{F})}}|^2\}^{1/2}$ *will be called the minimax estimation error of* $l(F)$.

Using the above results and definitions, formulate and prove the following statements.

Lemma 8.1. *Determination of the minimax estimate of* $l(F)$ *is equivalent to the problem of optimal control of the operator equation system*

$$L^+ z(t;u) = -C^* J_{H_0} u(t) \quad \text{on} \quad (a,b), \tag{8.4}$$

$$B_j^+(z(\cdot;u)) = 0 \quad j = \overline{1, 2n-m}, \tag{8.5}$$

$$\int_a^b Q^{-1}(l_0(t) + z(t;u))\overline{\psi_i(t)}dt + (Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;u))), \mathbf{S}^+(\psi_i))_{\mathbb{C}^m} = 0, \;\; i = \overline{1, m-r}, \tag{8.6}$$

with the cost function

$$I(u) = \int_a^b Q^{-1}(l_0(t) + z(t;u))\overline{(l_0(t) + z(t;u))}dt + (Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;u))), \mathbf{l} + \mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m} + (Q_0^{-1}u, u)_{H_0} \to \inf_{u \in V}, \tag{8.7}$$

where

$$V = \{u \in H_0 : \int_a^b C^* J_{H_0} u(t) \overline{\varphi_0(t)} dt = 0 \quad \forall \varphi_0 \in N(A_B)\}.$$

Proof. Note first that set V is is nonempty because any element $\tilde{u} \in H_0$ orthogonal to the subspace spanned over vectors $\{C\varphi_1, \ldots, C\varphi_{n-r}\}$ belongs to V and at any fixed $u \in V$function $z(t;u)$ is uniquely determined from equations (8.4)–(8.6). Indeed, condition $u \in V$ coincides, by virtue of (7.11), with the solvability condition of problem (8.4)–(8.5); let $z_0(t;u) \in W_2^n(a,b)$ be a solution to this problem. Then the function

$$z(t;u) := z_0(t;u) + \sum_{i=1}^{m-r} c_i \psi_i(t), \tag{8.8}$$

also satisfies (8.4)–(8.5) for any $c_i, \in \mathbb{C}^1$, $i = \overline{1, m-r)}$. Let us prove that coefficients c_i, $i = \overline{1, m-r}$, can be chosen so that this function would also satisfy condition (8.6). Substituting expression (8.8) for $z(t;u)$ into (8.6), we obtain a linear algebraic system of $m-r$ equations with $m-r$ unknowns $c_1, \ldots, c_{m-r}$:

$$\sum_{i=1}^{m-r} a_{ij} c_i = b_j(u), \quad j = 1, \ldots, m-r; \tag{8.9}$$

its matrix $[a_{ij}]_{i,j=1}^{m-r}$ whose elements a_{ij} are determined according to (7.27) is positive definite, thus $\det[a_{ij}] \neq 0$ (see p. 47) and elements $b_j(u)$ are determined from

$$b_j(u) = -\int_a^b Q^{-1}(l_0(t) + z_0(t;u))\overline{\psi_j(t)}dt$$
$$- (Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z_0(\cdot;u))), \mathbf{S}^+(\psi_j))_{\mathbb{C}^m}, \quad j = 1, \ldots, m-r. \tag{8.10}$$

Therefore this system has unique solution $c_1, \ldots, c_{m-r}$ and problem (8.4)–(8.6) is uniquely solvable.

Next, writing a solution $\tilde{\varphi}$ to problem (7.1), (7.2) in the form $\tilde{\varphi} = \tilde{\varphi}_\perp + \varphi_0$, where $\tilde{\varphi}_0$ and $\tilde{\varphi}_\perp$ are introduced on p. 48, and using the formula

$$(\tilde{y}, u)_{H_0} = (C\tilde{\varphi}, u)_{H_0} + (\tilde{\eta}, u)_{H_0} = (C(\tilde{\varphi}_\perp + \varphi_0), u)_{H_0} + (\tilde{\eta}, u)_{H_0}$$

$$=< C(\tilde{\varphi}_\perp + \varphi_0), J_{H_0}u >_{H_0 \times H_0'} + (\tilde{\eta}, u)_{H_0} = \int_a^b (\tilde{\varphi}_\perp(t) + \varphi_0(t))\overline{C^* J_{H_0} u(t)}\, dt + (\tilde{\eta}, u)_{H_0}$$

$$= \int_a^b \tilde{\varphi}_\perp(t)\overline{C^* J_{H_0} u(t)}\, dt + \int_a^b \varphi_0(t)\overline{C^* J_{H_0} u(t)}\, dt + (\tilde{\eta}, u)_{H_0},$$

for arbitrary $u \in H_0$, we have

$$l(\tilde{F}) - \widehat{l(\tilde{F})} = \int_a^b \overline{l_0(t)}\tilde{f}(t)\, dt + \sum_{j=1}^m \overline{l_j}\tilde{\alpha}_j - (\tilde{y}, u)_{H_0} - c$$

$$= \int_a^b \overline{l_0(t)}\tilde{f}(t)\, dt + \sum_{j=1}^m \overline{l_j}\tilde{\alpha}_j - \int_a^b \tilde{\varphi}_\perp(t)\overline{C^* J_{H_0} u(t)}\, dt$$

$$- \int_a^b \varphi_0(t)\overline{C^* J_{H_0} u(t)}\, dt - (\tilde{\eta}, u)_{H_0} - c.$$

The latter implies

$$M\left|l(\tilde{F}) - \widehat{l(\tilde{F})}\right|^2 = \left|\int_a^b \overline{l_0(t)}\tilde{f}(t)\, dt + \sum_{j=1}^m \overline{l_j}\tilde{\alpha}_j\right.$$

$$\left. - \int_a^b \tilde{\varphi}_\perp(t)(t)\overline{C^* J_{H_0} u(t)}\, dt - \int_a^b \varphi_0(t)\overline{C^* J_{H_0} u(t)}\, dt - c\right|^2 + M|(\tilde{\eta}, u)_{H_0}|^2. \tag{8.11}$$

Since function $\varphi_0(t)$ under the integral sign of the last term is an arbitrary element of space $N(A_B)$, quantity $\mathbf{M}|l(\tilde F)-\widehat{l(\tilde F)}|^2$ takes all values from $-\infty$ to $+\infty$. This quantity is finite if

$$\int_a^b \varphi_0(t)\overline{C^* J_{H_0} u(t)}\,dt = 0,$$

which is a necessary condition which holds if and only if $u\in V$. Assuming now that $u\in V$ and taking into account (8.4)–(8.6) and (7.5), we obtain the following representation for the expression under the sign of absolute value in (8.11)

$$\int_a^b \overline{l_0(t)}\tilde f(t)\,dt+\sum_{j=1}^m \overline{l_j}\tilde\alpha_j-\int_a^b \tilde\varphi_\perp(t)\overline{C^* J_{H_0} u(t)}\,dt - c$$

$$=\int_a^b \overline{l_0(t)}\tilde f(t)\,dt+\sum_{j=1}^m \overline{l_j}\tilde\alpha_j+\int_a^b \tilde\varphi_\perp(t)\overline{L^+ z(t;u)}\,dt - c$$

$$=\int_a^b \overline{l_0(t)}\tilde f(t)\,dt+\sum_{j=1}^m \overline{l_j}\tilde\alpha_j+\int_a^b L\tilde\varphi_\perp(t)\overline{z(t;u)}\,dt+\sum_{j=1}^m B_j(\tilde\varphi_\perp)\overline{S_j^+(z(\cdot;u))} - c$$

$$=\int_a^b \overline{l_0(t)}\tilde f(t)\,dt+\sum_{j=1}^m \overline{l_j}\tilde\alpha_j+\int_a^b \tilde f(t)\overline{z(t;u)}\,dt+\sum_{j=1}^m \tilde\alpha_j\overline{S_j^+(z(\cdot;u))} - c$$

$$=\int_a^b \overline{(l_0(t)+z(t;u))}\tilde f(t)\,dt+\sum_{j=1}^m \overline{(l_j+S_j^+(z(\cdot;u)))}\tilde\alpha_j - c$$

$$=(\tilde f, l_0+z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c.$$

The latter equality in combination with (8.11) yields

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,\tilde\eta\in G_1} M|l(\tilde F)-\widehat{l(\tilde F)}|^2 =$$

$$=\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0}\left|(\tilde f, l_0+z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\right|^2+\sup_{\tilde\eta\in G_1} M|(\tilde\eta,u)_{H_0}|^2. \tag{8.12}$$

To calculate the first term on the right-hand side of (8.12) use the generalized Cauchy–Bunyakovsky inequality ([10], p. 196) and (7.16) to obtain

$$\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,}\left|(\tilde f, l_0+z(\cdot;u))_{L^2(a,b)}+(\tilde\alpha,\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\right|^2$$

$$=\inf_{c\in\mathbb{C}}\sup_{\tilde F\in G_0,}\Big|\overline{(l_0+z(\cdot;u),\tilde f-f_0)_{L^2(a,b)}}+\overline{(\mathbf{l}+\mathbf{S}^+(z(\cdot;u)),\tilde\alpha-\alpha^{(0)})_{\mathbb{C}^m}}$$

$$+\ (f_0, l_0+z(\cdot;u))_{L^2(a,b)}+(\alpha^{(0)},\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\Big|^2$$

$$\le\big\{(Q^{-1}(l_0+z(\cdot;u)), l_0+z(\cdot;u))_{L^2(a,b)}$$
$$+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\big\}$$
$$\times\Big\{(Q(\tilde f-f^{(0)}),\tilde f-f^{(0)})_{L^2(a,b)}+(Q_1(\tilde\alpha-\alpha^{(0)}),\tilde\alpha-\alpha^{(0)})_{\mathbb{C}^m}\Big\}$$
$$\le\big\{(Q^{-1}(l_0+z(\cdot;u)), l_0+z(\cdot;u))_{L^2(a,b)}$$

$$+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\big\}. \tag{8.13}$$

The direct substitution shows that inequality (8.13) turns to equality at $\tilde{F}=(\tilde{f}(\cdot),\tilde{\alpha})=\tilde{F}^{(0)}:=(\tilde{f}^{(0)}(\cdot),\tilde{\alpha}^{(0)})=\left(\tilde{f}^{(0)}(\cdot),(\tilde{\alpha}_1^{(0)},\ldots,\ \tilde{\alpha}_m^{(0)})^T\right)\in L^2(a,b)\times\mathbb{C}^m$, where

$$\tilde{f}^{(0)}(t):=\frac{1}{d}Q^{-1}(l_0(t)+z(t,u))+f_0(t),$$

$$\tilde{\alpha}_i^{(0)}:=\frac{1}{d}Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_i+\alpha_i^{(0)},\, i=\overline{1,m},$$

$$d=\Big((Q^{-1}(l_0+z(\cdot;u)),l_0+z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\Big)^{1/2},$$

and $Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_j$ is the jth component of vector $Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))\in\mathbb{C}^m$. Element $\tilde{F}^{(0)}\in G_0$ because it obviously satisfies condition (7.16) and from the following chain of equalities

$$(\tilde{f}^{(0)},\psi_0)_{L^2(a,b)}+\sum_{i=1}^m\tilde{\alpha}_i^{(0)}\overline{S_i^+(\psi_0)}=$$

$$=\Big(Q^{-1}(l_0+z(\cdot;u)),l_0+z(\cdot;u))_{L^2(a,b)}+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}\Big)^{-1/2}$$
$$\times\Big(Q^{-1}(l_0+z(\cdot;u)),\psi_0)_{L^2(a,b)}+\sum_{i=1}^m Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_i\overline{S_i^+(\psi_0)}\Big)$$
$$+(f^{(0)},\psi_0)_{L^2(a,b)}+\sum_{i=1}^m\alpha_i^{(0)}\overline{S_i^+(\psi_0)}=0\quad\forall\psi_0\in N(A^+_{B^+})$$

it follows that this element also satisfies, in line with (8.6), condition (7.15). Therefore,

$$\inf_{c\in\mathbb{C}}\sup_{\tilde{F}\in G_0}\left|(\tilde{f},l_0+z(\cdot;u))_{L^2(a,b)}+(\tilde{\alpha},\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}-c\right|^2$$
$$=\int_a^b Q^{-1}(l_0(t)+z(t;u))\overline{(l_0(t)+z(t;u))}dt$$
$$+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))))_{\mathbb{C}^m} \tag{8.14}$$

at $c=\int_a^b\overline{(l_0(t)+z(t;u))}f^{(0)}(t)dt+(\alpha^{(0)},\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}$. For the second term on the right-hand side of (8.12), we have proved (see p. 50) that

$$\sup_{\tilde{\eta}\in G_1}M|(u,\tilde{\eta})_{H_0}|^2=(Q_0^{-1}u,u)_{H_0}. \tag{8.15}$$

The statement of Lemma 8.1 follows now from (8.12), (8.14), and (8.15). □

Theorem 8.1. *The minimax estimate of $l(F)$ can be represented as*

$$\widehat{\widehat{l(F)}}=(y,\hat{u})_{H_0}+\hat{c}, \tag{8.16}$$

where

$$\hat{u}=Q_0Cp,\quad\hat{c}=\int_a^b\overline{(l_0(t)+z(t))}f^{(0)}(t)dt+\sum_{i=1}^m\overline{(l_i+S_i^+(z))}\alpha_i^{(0)}, \tag{8.17}$$

and functions $p(t)$ and $z(t)$ are determined from the operator equation system

$$L^+z(t)=-C^*J_{H_0}Q_0Cp(t)\quad on\quad(a,b), \tag{8.18}$$

$$B_j^+(z)=0, \quad j=\overline{1,2n-m}, \tag{8.19}$$

$$\int_a^b Q^{-1}(l_0(t)+z(t))\overline{\psi_i(t)}\,dt+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0, \quad i=\overline{1,m-r}, \tag{8.20}$$

$$Lp(t)=Q^{-1}(l_0(t)+z(t)) \quad on \quad (a,b), \tag{8.21}$$

$$B_j(p)=Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_j, \quad j=\overline{1,m}, \tag{8.22}$$

$$\int_a^b C^*J_{H_0}Q_0Cp(t)\overline{\varphi_i(t)}dt=0,\ i=\overline{1,n-r}, \tag{8.23}$$

where $Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_j$ *denotes the* j*-a component of vector* $Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))\in\mathbb{C}^m$. *Problem* (8.18)–(8.23) *is uniquely solvable. The estimation error*

$$\sigma=l(\hat{\hat{P}})^{1/2}, \quad where \quad \hat{\hat{P}}=(Q^{-1}(l_0+z),(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_1,\ldots,Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_m)^T).$$

Proof. Show first that the solution to optimal control problem (8.4)–(8.7) is reduce to the solution of system (8.18)–(8.23). To this end, note that the form of functional (8.7) and the fact that $A^+_{B^+}$ is a Noether operator suggest that there is one and only one element $\hat{u}\in V$ at which the minumum of the functional is attained, $I(\hat{u})=\inf_{u\in V}I(u)$. Indeed, set $u=\bar{u}+v$ for an arbitrary $u\in V$ where $\bar{u}$ is a fixed element from V and $v=u-\bar{u}$. Then solution $z(t;u)$ to BVP (8.4)–(8.6) can be represented as

$$z(t;u)=z(t;\bar{u})+\tilde{z}(t;v), \tag{8.24}$$

where $z(t;\bar{u})$ and $\tilde{z}(t;v)$ are the unique solutions of this problem at $u=\bar{u}$ and $u=v$, $l_0(x)=0$, and $l_j=0$, $j=\overline{1,m}$; in addition, if v is an arbitrary element of V, then $u=\bar{u}+v$ is also an arbitrary element of this space.

Using expression (8.24) for $z(t;u)$ write functional $I(u)$ in the form

$$\begin{aligned}
I(u)&=\int_a^b Q^{-1}(l_0(t)+z(t;u))\overline{(l_0(t)+z(t;u))}dt\\
&+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;u))),\mathbf{l}+\mathbf{S}^+(z(\cdot;u)))_{\mathbb{C}^m}+(Q_0^{-1}u,u)_{H_0}\\
&=\int_a^b Q^{-1}(l_0(t)+z(t;\bar{u})+\tilde{z}(t;v))\overline{(l_0(t)+z(t;\bar{u})+\tilde{z}(t;v))}dt\\
&+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;\bar{u})+\tilde{z}(\cdot;v))),\mathbf{l}+\mathbf{S}^+(z(\cdot;\bar{u})+\tilde{z}(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}(\bar{u}+v),\bar{u}+v)_{H_0}\\
&=I(\bar{u})+\int_a^b Q^{-1}\tilde{z}(t;v)\overline{\tilde{z}(t;v)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)),\mathbf{S}^+(\tilde{z}(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}v,v)_{H_0}\\
&+2\mathrm{Re}\int_a^b Q^{-1}\tilde{z}(t;v)\overline{(l_0(t)+z(t;\bar{u}))}dt+2\mathrm{Re}(Q_1^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)),\mathbf{S}^+(l_0(\cdot)+z(\cdot;\bar{u}))_{\mathbb{C}^m}\\
&+2\mathrm{Re}(Q_0^{-1}v,\bar{u})_{H_0}=I(\bar{u})+\tilde{I}(v)+2\mathrm{Re}L(v),
\end{aligned} \tag{8.25}$$

where, due to the linearity and continuity of the mapping,

$$V\ni v\to(\tilde{z}(\cdot,v),(S_1^+(\tilde{z}(\cdot,v)),\ldots,S_m^+(\tilde{z}(\cdot,v)))^T)\in L^2(a,b)\times\mathbb{C}^m,$$

$$\tilde{I}(v)=\int_a^b Q^{-1}\tilde{z}(t;v)\overline{\tilde{z}(t;v)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)),\mathbf{S}^+(\tilde{z}(\cdot;v)))_{\mathbb{C}^m}+(Q_0^{-1}v,v)_{H_0} \tag{8.26}$$

is a linear quadratic functional in V associated with the continuous semi-bilinear form

$$\pi(v,w)=\int_a^b Q^{-1}\tilde{z}(t;v)\overline{\tilde{z}(t;w)}dt+(Q_1^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)),\mathbf{S}^+(\tilde{z}(\cdot;w)))_{\mathbb{C}^m}+(Q_0^{-1}v,w)_{H_0} \tag{8.27}$$

on $V \times V$, ; this functional satisfies

$$\tilde{I}(v) \geq c\|v\|_{H_0} \ \forall v \in V, \ c = \text{const}, \tag{8.28}$$

and

$$L(v) = \int_a^b Q^{-1}\tilde{z}(t;v)\overline{(l_0(t) + z(t;\bar{u}))}dt$$
$$+(Q_1^{-1}\mathbf{S}^+(\tilde{z}(\cdot;v)), \mathbf{S}^+(l_0(\cdot) + z(\cdot;\bar{u}))_{\mathbb{C}^m} + (Q_0^{-1}v, \bar{u})_{H_0} \tag{8.29}$$

is a linear continuous functional[11] in V.

Consequently (see Remark 1.1 to Theorem 1.1 in [3]), there exists the unique element $\hat{v} \in V$ (depending on $\bar{u}$) such that

$$I_V(\hat{v}) = \inf_{v \in V} I_V(v), = \inf_{v \in V} I(\bar{u} + v) = \inf_{u - \bar{u} \in V} I(u) = \inf_{u \in \bar{u} + V} I(u) = \inf_{u \in V} I(u).$$

Setting $\hat{u} = \bar{u} + \hat{v}$ and taking into account that

$$I_V(\hat{v}) = I(\bar{u} + \hat{v}) = I(\hat{u}),$$

we conclude that there is one and only one element $\hat{u} \in V$ admitting the representation $\hat{u} = \bar{u} + \hat{v}$ at which the minimum of functional $I(u)$ is attained for $u \in V$.

Therefore, for any $\tau \in R$ and $v \in V$,

$$\frac{d}{dt}I(\hat{u} + \tau v)\Big|_{\tau=0} = 0 \quad \text{and} \quad \frac{d}{dt}I(\hat{u} + i\tau v)\Big|_{\tau=0} = 0, \tag{8.30}$$

where $i = \sqrt{-1}$. Since $z(t;\hat{u} + \tau v) = z(t;\hat{u}) + \tau\tilde{z}(t;v)$, where $\tilde{z}(t;v)$ is the unique solution to BVP (8.4)–(8.6) at $u = v$, $l_0 = 0$, and $l_j = 0$, $j = \overline{1,m}$, we can use the first relationship in (8.30) to obtain

$$0 = \frac{1}{2}\frac{d}{d\tau}I(\hat{u} + \tau v)|_{\tau=0}$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau}\Big\{\Big[(Q^{-1}(l_0 + z(\cdot;\hat{u} + \tau v)), l_0 + z(\cdot;\hat{u} + \tau v))_{L^2(a,b)} - (Q^{-1}(l_0 + z(\cdot;\hat{u})), l_0 + z(\cdot;\hat{u}))_{L^2(a,b)}\Big]$$
$$+\Big[(Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u} + \tau v))), \mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u} + \tau v)))_{\mathbb{C}^m} - (Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u}))), \mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u})))_{\mathbb{C}^m}\Big]$$
$$+ \left[Q_0^{-1}(\hat{u} + \tau v), \hat{u} + \tau v)_{H_0} - (Q_0^{-1}\hat{u}, \hat{u})_{H_0}\right]\Big\}$$
$$= \mathrm{Re}\{(Q^{-1}(l_0 + z(\cdot;\hat{u})), \tilde{z}(\cdot;v))_{L^2(a,b)}$$
$$+(Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u}))), \mathbf{S}^+(\tilde{z}(\cdot;v)))_{\mathbb{C}^m} + (Q_0^{-1}\hat{u}, v)_{H_0}\}.$$

Next, since $z(t;\hat{u} + i\tau v) = z(t;\hat{u}) + i\tau\tilde{z}(t;v)$, we have

$$0 = \frac{1}{2}\frac{d}{d\tau}I(\hat{u} + i\tau v)|_{\tau=0}$$

$$= \mathrm{Im}\{(Q^{-1}(l_0 + z(\cdot;\hat{u})), \tilde{z}(\cdot;v))_{L^2(a,b)}$$
$$+(Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z(\cdot;\hat{u}))), \mathbf{S}^+(\tilde{z}(\cdot;v)))_{\mathbb{C}^m} + (Q_0^{-1}\hat{u}, v)_{H_0}\},$$

which yields

$$(Q^{-1}(l_0 + z(\cdot;\hat{u})), \tilde{z}(\cdot;v))_{L^2(a,b)}$$

[11] The continuity of form (8.27) is proved on p. 55, and the continuity of linear functional (8.29) can be proved similarly to the case of functional (7.80).

$$+\sum_{j=1}^{m} Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;\hat{u})))_j\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}+(Q_0^{-1}\hat{u},v)_{H_0}=0. \tag{8.31}$$

Let $p(t)$ be a solution[12] to the BVP

$$Lp(t)=Q^{-1}(l_0(t)+z(t;\hat{u})) \quad \text{on} \quad (a,b), \tag{8.32}$$

$$B_j(p)=Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot,\hat{u})))_j, \quad j=\overline{1,m}. \tag{8.33}$$

Then, taking into consideration Green's formula (7.5), we can transform the sum of the first two terms on the left-hand side of (8.31) as follows:

$$(Q^{-1}(l_0+z(\cdot;\hat{u})),\tilde{z}(\cdot;v))_{L^2(a,b)}+\sum_{j=1}^{m} Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z(\cdot;\hat{u})))_j\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}$$

$$=\int_a^b Lp(t)\overline{\tilde{z}(t;v)}\,dt+\sum_{j=1}^{m} B_j(p)\overline{\mathbf{S}_j^+(\tilde{z}(\cdot;v))}=\int_a^b p(t)\overline{L^+\tilde{z}(t;v)}\,dt$$

$$=-\int_a^b p(t)\overline{C^*J_{H_0}v(t)}\,dt=-<Cp,J_{H_0}v>_{H_0\times H_0'}=-(Cp,v)_{H_0}.$$

From the latter equality and relationship (8.31), it follows that

$$(Q_0^{-1}\hat{u}-Cp,v)_{H_0}=0 \tag{8.34}$$

for any $v\in V$. Next, repeating the reasoning contained in the proof of Theorem 2.1 on p. 57, we see that among all solutions to problem (8.32), (8.33), there is one and only one solution $p(t)$ for which

$$Q_0^{-1}\hat{u}-Cp\in V. \tag{8.35}$$

Setting in (8.34) $v=Q_0^{-1}\hat{u}-Cp$ we obtain $Q_0^{-1}\hat{u}-Cp=0$, so that $\hat{u}=Q_0Cp$. Substituting this into the equality $\int_a^b C^*J_{H_0}\hat{u}(t)\overline{\varphi_0(t)}dt=0$ and denoting $z(t;\hat{u})=:z(t)$, we see that $z(t)$ and $p(t)$ satisfy system (8.18)–(8.23); the unique solvability of this system follows from the uniqueness of element $\hat{u}$.

Show now that $\sigma(\hat{u},\hat{c})=l(\hat{\hat{P}})^{1/2}$; then the estimate $\sigma(F)\le l(\hat{\hat{P}})^{1/2}$ will be proved. Substituting $\hat{u}=Q_0Cp$ into the expression for $I(\hat{u})$ and taking into account that $z(t)=z(t;\hat{u})$, we have

$$I(\hat{u})=\int_a^b Q^{-1}(l_0(t)+z(t))\overline{(l_0(t)+z(t))}\,dt+(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z)),\mathbf{l}+\mathbf{S}^+(z))_{C^m}$$

$$+(Cp,Q_0Cp)_{H_0}=\int_a^b Lp(t)\overline{z(t)}\,dt+\int_a^b \overline{l_0(t)}Q^{-1}(l_0(t)+z(t))\,dt$$

$$+\sum_{j=1}^{m} Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_j(\overline{l_j+S_j^+(z)})+(Cp,Q_0Cp)_{H_0}$$

$$=\int_a^b Lp(t)\overline{z(t)}\,dt+\int_a^b \overline{l_0(t)}Q^{-1}(l_0(t)+z(t))\,dt$$

$$+\sum_{j=1}^{m} B_j(p)\overline{S_j^+(z)}+\sum_{j=1}^{m}\overline{l_j}Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_j+(Cp,Q_0Cp)_{H_0}$$

$$=\int_a^b p(t)\overline{L^+(t)}\,dt+\sum_{j=1}^{2n-m} S_j(p)\overline{B_j^+(z)}+\int_a^b \overline{l_0(t)}Q^{-1}(l_0(t)+z(t))\,dt$$

[12]Formula (8.20) coincides with the solvability condition for this problem by virtue of (7.10).

$$+\sum_{j=1}^{m}\overline{l_j}Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_j+(Cp,Q_0Cp)_{H_0}. \tag{8.36}$$

For the first term in (8.36) we have

$$\int_a^b p(t)\overline{L^+(t)}\,dt=-\int_a^b p(t)(\overline{C^*I_{H_0}Q_0Cp)(t)})\,dt=-<Cp,I_{H_0}Q_0Cp>_{H_0\times H_0'}.$$

From the latter equality and (8.36), it follows that $I(\hat{u})=l(\hat{\hat{P}})$, where

$$\hat{\hat{P}}=(Q^{-1}(l_0+z),(Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_1,\ldots,Q_1^{-1}(\mathbf{l}+\mathbf{S}^+(z))_m)^T).$$

The proof of the theorem is completed. □

Another representation for the minimax estimate is given by the following theorem.

Theorem 8.2. *The minimax estimate of $l(F)$ has the form*

$$\widehat{\widehat{l(F)}}=l(\hat{F}), \tag{8.37}$$

where

$$\hat{F}=(\hat{f}(\cdot),(\hat{\alpha}_1,\ldots,\hat{\alpha}_m)^T),\ \ \hat{f}(t)=Q^{-1}\hat{p}(t)+f^{(0)}(t),$$
$$\hat{\alpha}_j=Q_1^{-1}\mathbf{S}^+(\hat{p})_j+\alpha_j^{(0)},\ j=\overline{1,m},$$

and function $\hat{p}$ is determined from the solution to the problem

$$L^+\hat{p}(t)=C^*J_{H_0}Q_0(y-C\hat{\varphi})(t)\quad \textit{on}\quad (a,b), \tag{8.38}$$

$$B_j^+(\hat{p})=0,\quad j=\overline{1,2n-m}, \tag{8.39}$$

$$\int_a^b Q^{-1}\hat{p}(t)\overline{\psi_i(t)}\,dt+(Q_1^{-1}\mathbf{S}^+(\hat{p}),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0,\quad i=\overline{1,m-r}, \tag{8.40}$$

$$L\hat{\varphi}(t)=Q^{-1}\hat{p}(t)+f^{(0)}(t)\quad \textit{on}\quad (a,b), \tag{8.41}$$

$$B_j(\hat{\varphi})=Q_1^{-1}\mathbf{S}^+(\hat{p})_j+\alpha_j^{(0)},\quad j=\overline{1,m}, \tag{8.42}$$

$$\int_a^b C^*J_{H_0}Q_0\,(y-C\hat{\varphi})\,(t)\overline{\varphi_i(t)}\,dt=0,\quad i=\overline{1,n-r}, \tag{8.43}$$

where $\mathbf{S}^+(\hat{p}):=(S_1^+(\hat{p}),\ldots,\ S_m^+(\hat{p}))^T\in\mathbb{C}^m$ is the vector with components $S_j^+(\hat{p})$, $j=\overline{1,m}$. Problem (8.38)–(8.43) *is uniquely solvable.*

Proof. Introduce the problem of optimal control of the equation system

$$L^+\hat{p}(t;u)=-(C^*J_{H_0}u)(t)+(C^*J_{H_0}Q_0(y)(t)\quad \text{on}\quad (a,b), \tag{8.44}$$

$$B_j^+(\hat{p}(\cdot;u))=0\quad (j=1,\ldots,m), \tag{8.45}$$

$$\int_a^b Q^{-1}\hat{p}(t;u)\overline{\psi_i(t)}dt+(Q_1^{-1}\mathbf{S}^+(\hat{p}(\cdot;u)),\mathbf{S}^+(\psi_i))_{\mathbb{C}^m}=0,\quad i=\overline{1,m-r}, \tag{8.46}$$

with the cost function

$$I(u)=\int_a^b Q^{-1}(\hat{p}(t;u)+Qf^{(0)}(t))\overline{(\hat{p}(t;u)+Qf^{(0)}(t))}dt$$
$$+(Q_1^{-1}(\mathbf{S}^+(\hat{p}(\cdot;u))+Q_1\alpha^{(0)}),\mathbf{S}^+(\hat{p}(\cdot;u))+Q_1\alpha^{(0)})_{\mathbb{C}^m}+(Q_0^{-1}u,u)_{H_0}\to\inf_{u\in U}, \tag{8.47}$$

where

$$U = \{u \in H_0 : \int_a^b (C^* J_{H_0} Q_0 (y(t) - C^* J_{H_0} u(t)) \overline{\varphi_0(t)} dt = 0\}$$

for any solutions $\varphi_0(t)$ of homogeneous problem (7.1), (7.2).

The form of functional $I(u)$ and reasoning contained in the proof of Theorem 8.1 suggest the existence of unique element $\hat{u} \in U$ such that

$$I(\hat{u}) = \inf_{u \in U} I(u).$$

Next, funding $\hat{\varphi}(t)$ as the unique solution to the problem

$$L\hat{\varphi}(t) = Q^{-1}\hat{p}(t;\hat{u}) + f^{(0)}(t) \quad \text{on} \quad (a,b),$$

$$B_j(\hat{\varphi}) = Q_1^{-1}\mathbf{S}^+(\hat{p}(\cdot;\hat{u})_j + \alpha_j^{(0)}, \quad j = \overline{1,m},$$

$$\int_a^b C^* J_{H_0} Q_0 \left(y - C\hat{\varphi}\right)(t) \overline{\varphi_i(t)}\, dt = 0, \quad i = \overline{1, n-r},$$

and repeating the proof of Theorem 8.1, we conclude, taking into consideration the notation $\hat{p}(t) = \hat{p}(t;\hat{u})$, that problem (8.38)–(8.43) is uniquely solvable.

Let us prove the validity of the representation $\widehat{\widehat{l(F)}} = l(\hat{F})$. Substituting expressions (8.17) for $\hat{u}$ and $\hat{c}$ into (8.16) and using (8.38)–(8.40), we obtain

$$\widehat{\widehat{l(F)}} = (y, \hat{u})_{H_0} + \hat{c} = (y, Q_0 C p)_{H_0} + \hat{c} = \overline{(Cp, Q_0 y)_{H_0}} + \hat{c}$$

$$= \overline{< Cp, J_{H_0} Q_0 y >_{H_0 \times H_0'}} = \overline{(p, C^* J_{H_0} Q_0 y)_{L^2(a,b)}} + \hat{c}$$

$$= \overline{\int_a^b p(t) \overline{C^* J_{H_0} Q_0 y(t)}\, dt} + \hat{c} = \overline{\int_a^b p(t) \overline{L^+ \hat{p}(t)}\, dt} + \int_a^b \overline{(l_0(t) + z(t))} f^{(0)}(t) dt$$

$$+ \sum_{j=1}^m \overline{(l_j + S_j^+(z))} \alpha_j^{(0)} + \overline{\int_a^b p(t) \overline{C^* J_{H_0} Q_0 C \hat{\varphi}(t)}\, dt}. \tag{8.48}$$

Transform the sum of the first three terms on the right-hand side of this equality using Green's formula (7.5) and taking into account equalities (8.18)–(8.23) and (8.41)–(8.43). As a result, we obtain

$$\overline{\int_a^b p(t) \overline{L^+ \hat{p}(t)}\, dt} + \int_a^b \overline{(l_0(t) + z(t))} f^{(0)}(t) dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))} \alpha_j^{(0)}$$

$$= \overline{\int_a^b Lp(t) \overline{\hat{p}(t)}\, dt} + \sum_{j=1}^m \overline{B_j(p) \overline{S_j^+(\hat{p}(t))}}$$

$$+ \int_a^b \overline{(l_0(t) + z(t))} f^{(0)}(t) dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))} \alpha_j^{(0)}$$

$$= \overline{\int_a^b Q^{-1}(l_0(t) + z(t)) \overline{\hat{p}(t)}\, dt} + \sum_{j=1}^m \overline{Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z))_j \overline{S_j^+(\hat{p})}}$$

$$+ \int_a^b \overline{(l_0(t) + z(t))} f^{(0)}(t) dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))} \alpha_j^{(0)}$$

$$= \overline{\int_a^b Q^{-1}(l_0(t) + z(t)) \overline{\hat{p}(t)}\, dt} + \overline{(Q_1^{-1}(\mathbf{l} + \mathbf{S}^+(z), \mathbf{S}^+(\hat{p}))_{\mathbb{C}^m}}$$

$$+\int_a^b \overline{(l_0(t)+z(t))} f^{(0)}(t)dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))}\alpha_j^{(0)}$$

$$= \overline{\int_a^b (l_0(t)+z(t))\overline{Q^{-1}\hat{p}(t)}\,dt} + (\overline{\mathbf{l}+\mathbf{S}^+(z), Q_1^{-1}\mathbf{S}^+(\hat{p}))_{\mathbb{C}^m}}$$

$$+\int_a^b \overline{(l_0(t)+z(t))} f^{(0)}(t)dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))}\alpha_j^{(0)}$$

$$= \int_a^b \overline{(l_0(t)+z(t))} Q^{-1}\hat{p}(t)\,dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))\overline{(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j}}$$

$$+\int_a^b \overline{(l_0(t)+z(t))} f^{(0)}(t)dt + \sum_{j=1}^m \overline{(l_j + S_j^+(z))}\alpha_j^{(0)}$$

$$= l(\hat{F}) + \int_a^b \overline{z(t)} Q^{-1}\hat{p}(t)dt + \sum_{j=1}^m \overline{S_j^+(z)}(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j$$

$$+\int_a^b \overline{z(t)} f^{(0)}(t)dt + \sum_{j=1}^m \overline{S_j^+(z)}\alpha_j^{(0)}$$

$$= l(\hat{F}) + \int_a^b \overline{z(t)}(Q^{-1}\hat{p}(t) + f^{(0)}(t))dt + \sum_{j=1}^m \overline{S_j^+(z)}(Q_1^{-1}\mathbf{S}^+(\hat{p}))_j + \alpha_j^{(0)})$$

$$= l(\hat{F}) + \int_a^b L\hat{\varphi}(t)\overline{z(t)}dt + \sum_{j=1}^m B_j(\hat{\varphi})\overline{S_j^+(z)} = l(\hat{F}) + \int_a^b \hat{\varphi}(t)\overline{L^+z(t)}dt$$

$$= l(\hat{F}) - \int_a^b \hat{\varphi}(t)\overline{C^*J_{H_0}Q_0Cp(t)}dt = l(\hat{F}) - < C\hat{\varphi}, J_{H_0}Q_0Cp >_{H_0\times H_0'}$$

$$= l(\hat{F}) - (C\hat{\varphi}, Q_0Cp)_{H_0} = l(\hat{F}) - (Q_0C\hat{\varphi}, Cp)_{H_0} = l(\hat{F}) - \overline{(Cp, Q_0C\hat{\varphi})_{H_0}}$$

$$= l(\hat{F}) - \overline{< Cp, J_{H_0}Q_0C\hat{\varphi} >_{H_0\times H_0'}} = l(\hat{F}) - \overline{(p, C^*J_{H_0}Q_0C\hat{\varphi})_{L^2(a,b)}}$$

$$= l(\hat{F}) - \overline{\int_a^b p(t)\overline{C^*J_{H_0}Q_0C\hat{\varphi}(t)}dt}.$$

The resulting relationships together with equality (8.48) yield the sought-for representation. □

Corollary 8.1. *The function $\hat{f} = Q^{-1}\hat{p} + f^{(0)}$ and numbers $\hat{\alpha}_j = Q_1^{-1}\mathbf{S}^+(\hat{p})_j + \alpha_j^{(0)}$ can be taken as estimates of the right-hand sides f and α_j ($j = \overline{1,m}$) of equalities (7.1) and (7.2), respectively.*

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